Pricing Catastrophe Excess of Loss Reinsurance using Market Curves

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Abstract: What is a simple way to price a catastrophe excess of loss reinsurance program (Cat XL)? By simple we mean pricing a Cat XL with limited information. This paper presents pricing methods that only require the layer pricing of last year's Cat XL program and do not require any catastrophe modelling output.

The first method is to fit a power curve (i.e. a market curve) through the midpoints of the original Cat XL layers and then using that power curve to price the new program. This method has a history of actual use in the reinsurance market.

However, power curves have three key weaknesses and we therefore propose a new method. In this new method we propose a more sophisticated spline curve as the market curve, and unlike the power curve, layers are not represented by their midpoints, but rather by integrating from one endpoint to another. We show how this spline method resolves the three weaknesses of the power curve method.

Note:

An Excel workbook accompanies this paper. There are tabs numbered from #1 to #10. We invite the reader to follow along in the workbook as instructed in the paper so as to increase his or her understanding of the methods. In the workbook, cells that serve as user inputs are highlighted in green. The parameters of market curves (power curves and splines) and the outputs of those market curves are shown in blue.

There are three graphs presented in the workbook that correspond with the three graphs presented in this paper. Should the reader wish to use his or her own Cat XL program in the workbook the axes of the graphs may need to be modified.

1. MOTIVATION OF THE PROBLEM

To motivate the problem let's assume a catastrophe excess of loss reinsurance program (abbreviated in this paper as Cat XL) for a fictional insurance company called Island Insurance. Island Insurance writes property insurance exclusively on a small island with exposure to catastrophic perils such as hurricanes and earthquakes. The total insured value of all of Island Insurance's policies (abbreviated in this paper as TIV) adds up to $2.7 billion USD. Island protects itself from the catastrophic perils with a Cat XL program as follows:

Table 1 - Original Program				
Total Insured Value - TIV		2,700,000,000		
Layer i	**Limit - L**	**Deductible - D**	**ROL - R**	**C = L x R** **Cost - C**
Layer 1	5,000,000	5,000,000	20.70%	1,035,000
Layer 2	10,000,000	10,000,000	14.55%	1,455,000
Layer 3	30,000,000	20,000,000	10.20%	3,060,000
Layer 4	50,000,000	50,000,000	6.42%	3,210,000
Layer 5	55,000,000	100,000,000	3.75%	2,062,500
Total Program	**150,000,000**	**5,000,000**	**7.22%**	**10,822,500**

Some comments:

- ROL = Rate on Line = upfront cost of reinsurance layer / Limit of Layer
- Cost = Limit of Layer x ROL
- We ignore reinstatements by assuming that all layers are purchased with the same reinstatement conditions.
- The green cells are user inputs. We recommend that the reader follow along by opening the blank workbook that accompanies this paper, select tab #1 and fill in TIV = 2,700,000,000 in cell D3 and the appropriate Limits, Deductibles and ROLs in columns C, D and E. Note that only the first deductible is necessary in cell D6.

Let us now say that for the following year, Island's TIV went up from $2.7B to $3.0B and also they are restructuring the program into four layers: $7.5m xs $7.5m, $20m xs $15m, $50m xs $35m and $90m xs $85m (for a total program of $167.5m xs $7.5m, thus increasing their total limit from $150m to $167.5m and their retention from $5m to $7.5m).

The question that this paper attempts to answer is straightforward - what do we expect the new market ROLs to be for the new program layers if we don't have any addition information? The only information we have at our disposal is last year's Cat XL program and TIV (all given in Table 1) and this year's proposed Cat XL program and new TIV. We do not have catastrophe modelling information.

Although the TIV is changing year over year, we will otherwise be assuming "flat" renewal conditions:

- Underlying mix of business stays the same
- Geographic footprint stays the same
- Reinsurance market is neither hardening nor softening

What we want is a starting point for the Cat XL renewal.

2. CURRENT SOLUTION: FITTING A POWER CURVE

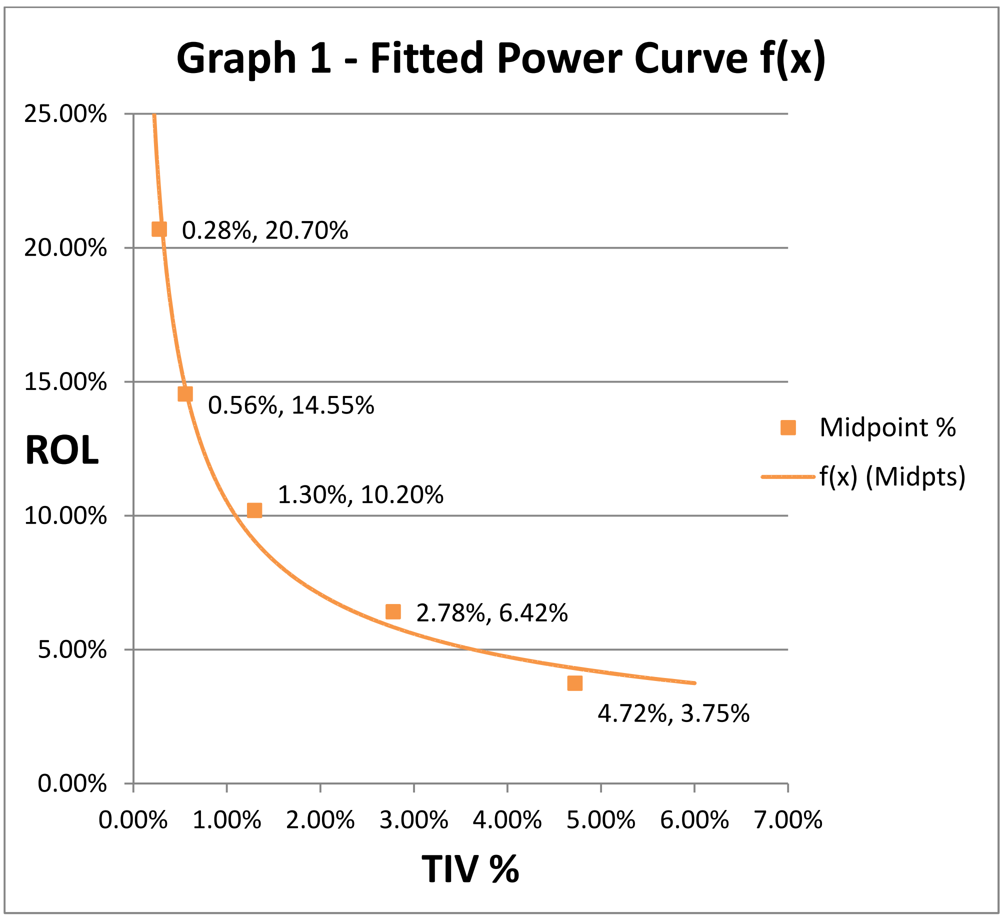
The current solution to the problem posed above is to fit a power curve through the midpoints of the original program layers. This method of fitting a power curve has been known to participants in the London reinsurance market since at least the early 1990s; however, no published document presenting this method has been found by the author of this paper.

In particular, the current solution is as follows: for each layer in the original program, calculate the midpoint of the layer as a % of the original TIV. These are the x values. By midpoint we are referring here to the arithmetic mean or simple average so that if the limit of the layer is L and the deductible of the layer is D, then $x = \frac{AVG(D, D+L)}{TIV}$.

The y values are the ROLs of the layers. Let $f(x) = y = a * x^{-b}$ be a power curve through the points (x, y). We take the logarithm of both sides, and get $\ln(y) = \ln(a) - b * \ln(x)$. ln(y) and ln(x) are thus related linearly, and we calculate a and b to minimize the SSE between the left hand side and the right hand side of the equation.

For Island Insurance, we calculate a = 0.00742 and b = 0.57591. These parameters can be found in blue on tab #2. The actual regression formulas can be found in the hidden columns K and L.

Graphically the power curve looks as follows:



We now use f(x) to price out the new program. In tab #4 we can enter in the new TIV of 3,000,000,000 in cell C3 and the new layering (four new layers) in columns C and D. The new midpoints as a % of the new TIV are calculated in column E, and f(x) is applied to these midpoints to get the new ROLs in column F.

The result is as follows:

Table 2 - New Program Layering: Priced using Power Curve f(x)					
New TIV	3,000,000,000				
			AVG(D, D+L) / TIV	**f(MP)**	**L x ROL_1**
Layer i	**Limit - L**	**Deductible - D**	**Midpoint % - MP**	**ROL_1**	**$Cost_1$**
Layer 1	7,500,000	7,500,000	0.38%	18.51%	1,388,155
Layer 2	20,000,000	15,000,000	0.83%	11.69%	2,337,163
Layer 3	50,000,000	35,000,000	2.00%	7.06%	3,529,088
Layer 4	90,000,000	85,000,000	4.33%	4.52%	4,069,582
Total	**167,500,000**	**7,500,000**		**6.76%**	**11,323,987**

Example Calculation 1 –

Let's calculate the cost of layer #2 of the new program. This layer is \$20m xs \$15m. Given a TIV of \$3B, the midpoint % is x = $\frac{(\$35m+\$15m)/2}{\$3b}$ = 0.833%. Then we calculate the ROL = $f(x) = a * x^{-b} = 0.00742 \times (0.833\%)^{-0.57591}$ = 11.69%. So the cost of the layer = ROL x Limit = 11.69% x \$20m = \$2.34m. (This calculation can be found in cells E7 and F7 on tab #4.)

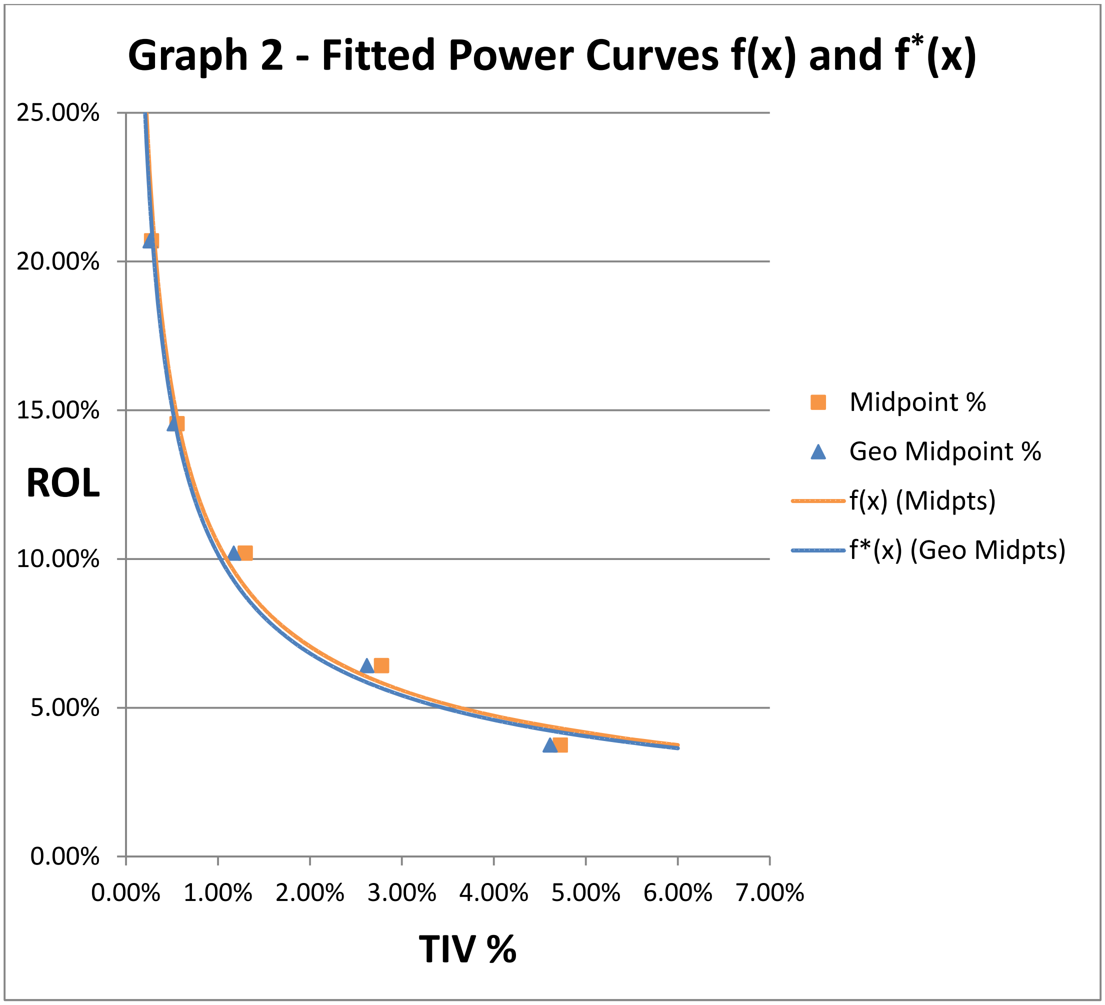
3. POWER CURVE USING GEOMETRIC MIDPOINTS

The solution above we might refer to as the power curve method using midpoints. However, instead of arithmetic midpoints (arithmetic mean) we could also take the geometric mean of each layer to get slightly different results.

Similar to before, for each layer in the original program we calculate the geometric midpoint of the layer as a % of the original TIV. These are the new x values. For a layer limit L and a layer deductible D, we have $x = \frac{\sqrt{D*(D+L)}}{TIV}$. The y values are the same as before – the ROLs of the layers. Let $f^*(x) = y = a * x^{-b}$ be a new power curve through the new points (x, y). We can once again calculate a and b after taking the logarithm of both sides and solving the linear regression.

For Island Insurance, we calculate a = 0.00727 and b = 0.57264. These parameters can be found in blue on tab #5. The actual regression formulas can be found in the hidden columns K and L.

Graphically we can look at both power curves side by side:



We now use $f^*(x)$ to price out the new program. In tab #7 the new TIV of 3,000,000,000 and the new Cat XL program structure are passed over from tab #4 and the calculated ROLs using both f(x) and $f^*(x)$ are shown in columns G and H respectively.

The result is as follows:

Table 3 - New Program Layering: Priced using Power Curves f(x) and $f^*(x)$						
New TIV	3,000,000,000					
			AVG(D, D+L) / TIV	**SQRT[D x (D+L)] / TIV**	**f(MP)**	**f^*(GMP)**
Layer i	**Limit - L**	**Deductible - D**	**Midpoint % - MP**	**Geo Midpoint % - GMP**	**ROL_1**	**ROL_2**
Layer 1	7,500,000	7,500,000	0.38%	0.35%	18.51%	18.42%
Layer 2	20,000,000	15,000,000	0.83%	0.76%	11.69%	11.85%
Layer 3	50,000,000	35,000,000	2.00%	1.82%	7.06%	7.21%
Layer 4	90,000,000	85,000,000	4.33%	4.07%	4.52%	4.55%
Total	**167,500,000**	**7,500,000**			**6.76%**	**6.84%**

Example Calculation 2 –

Let's calculate the cost of layer #2 of the new program using $f^*(x)$. This layer is \$20m xs \$15m. Given a TIV of \$3B, the geometric midpoint % is x = $\frac{\sqrt{\$15m * \$35m}}{\$3b}$ = 0.764%. Then we calculate the ROL = $f^*(x) = a * x^{-b} = 0.00727 \times (0.764\%)^{-0.57264}$ = 11.85%. So the cost of the layer = ROL x Limit = 11.85% x \$20m = \$2.37m. (This calculation can be found in cells F7 and H7 on tab #7.)

What happened when we used geometric midpoints? In *this* case, as we can see in Table 3, we have higher ROLs for Layers 2, 3 and 4 and a lower ROL for Layer 1. Overall the program pricing is higher at 6.84% ROL using geometric midpoints than 6.76% ROL using arithmetic midpoints. We stress that these pricing differences are for Island Insurance only, and the author has not found a general rule as to when geometric midpoints lead to higher prices than arithmetic midpoints and vice versa.

Having now looked at fitting power curves to both types of midpoints, we might now naturally ask, which type of midpoint is better? In the author's practice arithmetic midpoints are used first because they are simpler to understand, and geometric midpoints are used second as a complement to the arithmetic midpoints (if at all).

However, using geometric midpoints may have a theoretical justification. Note that the geometric midpoint of a given layer is always smaller than (to the left of) the arithmetic midpoint. Furthermore, since the power curve is a decreasing function, the "weighted" midpoint of a layer will also be to the left of the arithmetic midpoint.

4. WEAKNESSES OF POWER CURVES

Weakness #1 (Pricing of Original Program) – To see the first weakness of the power curve method let's price out the original Cat XL program for Island Insurance on f(x), which can be seen on tab #2:

Table 4 - Fit a Power Curve through Original ARITHMETIC Midpoints					Fitted ROLs		
Total Insured Value - TIV	2,700,000,000						
			AVG(D, D+L) / TIV		**r = f(MP)**	**c = L x r**	**(r - R) / R**
Layer i	**Limit - L**	**Deductible - D**	**Midpoint % - MP**	**ROL - R**	**ROL - r**	**Cost - c**	**Error %**
Layer 1	5,000,000	5,000,000	0.28%	20.70%	22.00%	1,100,036	6.3%
Layer 2	10,000,000	10,000,000	0.56%	14.55%	14.76%	1,475,949	1.4%
Layer 3	30,000,000	20,000,000	1.30%	10.20%	9.06%	2,718,141	-11.2%
Layer 4	50,000,000	50,000,000	2.78%	6.42%	5.84%	2,920,782	-9.0%
Layer 5	55,000,000	100,000,000	4.72%	3.75%	4.30%	2,366,871	14.8%
Total Program	**150,000,000**	**5,000,000**		**7.22%**	**7.05%**	**10,581,778**	**-2.2%**

Here we have the original TIV of \$2.7B and the original layering, yet when we apply f(x) to the midpoint %'s we get ROLs that are different from the original ROLs. In some cases the error % is high; for the third layer f(x) is underestimating the ROL by 11.2%, for the fifth layer f(x) is overestimating the ROL by 14.8%.

These errors can also be seen by looking at the power curve in Graph 1 – notice that the curve does not go precisely through the points (the actual ROLs), some are above the curve and some are below. Similar error %s can be found for $f^*(x)$ on tab #5.

Naturally, whatever pricing method we choose, we would want the new prices (the starting point) to be the same as the old prices if nothing has changed. The power curve method does not have this important desired property.

Weakness #2 (Non Uniqueness of Layers) – To see the second weakness of the power curve method let's consider the following four distinct layers (Limit L xs Deductible D):

- $1m xs $12m
- $5m xs $10m
- $10m xs $7.5m
- $15m xs $5m

Notice that the midpoint of each of the above layers is $12.5m (Midpoint = AVG(D, D+L) = D + L/2), meaning that under the power curve method (using arithmetic midpoints), each of the above layers would be assigned the same ROL under f(x). Many other layers could be generated.

While we might expect the ROLs for some of these layers to be similar or even the same, there is no reason to believe that *all* of these layers *must* have the same ROLs, as required by the power curve method, so we can consider this a weakness.

Weakness #3 (Unboundedness) – Notice that the power curves are unbounded. $f(x) = a * x^{-b}$ goes to infinity as x goes to 0. This means that if we use a power curve to price layers excess of 0 (i.e. Cat XL layers with no deductible), then the ROL of these layers will get arbitrarily large as the midpoint approaches 0. We will eventually have ROLs (e.g. 1,000%) that do not make sense.

5. PROPOSED SOLUTION: SPLINE CURVE

The power curves above allow the user to find the "market price" of a given Cat XL layer. Thus, in a more general sense, we might refer to these power curves as market curves, and we might also expect to find other, different market curves.

The new market curve proposed in this section is the use of a spline, fitted to the original Cat XL program.

Our first step is to re-envision the way the curves are used to calculate the premium cost of a layer. Instead of getting the midpoint % for the layer and calculating the ROL using f(x) or $f^*(x)$, as we have been doing with the power curve method, let's instead use integration, and integrate from one endpoint of the layer to the other:

Example Calculation 3 –

Let's calculate the cost of the same layer #2 in the new program, the layer $20m xs $15m, as we have in example calculations 1 and 2, but this time by integrating f(x) across the layer. First let's define the endpoints of the layer. Let LP_2 = $15m / $3b = 0.5% be the left endpoint as a % of TIV, and let RP_2 =$35m / $3b = 1.167% be the right endpoint as a % of TIV. Integrating f(x) from LP_2 to RP_2, we have cost = $\int_{LP_2}^{RP_2} a * x^{-b} dx = \frac{a}{-b+1} * \left(x^{-b+1} \Big|_{LP_2}^{RP_2} \right)$.

Plugging in a = 0.00742 and b = 0.57591, we get cost = $\frac{0.00742}{-0.57591+1} \times \left(1.167\%^{-0.57591+1} - 0.5\%^{-0.57591+1}\right)$ = 0.08%.

Since we are integrating across x, and x is expressed as a % of the TIV, this cost is also a % of the TIV. So the cost in dollars would be 0.08% x $3b = $2.40m. Finally, the ROL can be worked out as ROL = Cost / Limit = $2.40m / $20m = 12.0%.

Notice that the use of integration to calculate the layer costs automatically resolves weakness #2 of the power curve method. That is to say, layers with the same midpoints do not necessarily yield the same ROLs under integration. That is because layers are uniquely defined by their two endpoints, and since integration happens from one endpoint of a layer to the other endpoint, each layer has a unique integration.

The second step of the proposal is to pick a price curve that improves upon f(x). Let's call this new price curve g(x). We would want to pick a g(x) that has the following features:

- Resolves weakness #1 of the power curve. In other words, if we use g(x) to price the original program, we should get the original ROLs.
- The function should be bounded on the top and on the bottom. By bounding the function on the top, as it goes to 0, we resolve weakness #3. By bounding on the bottom, we have a chance to incorporate market knowledge that is external to the Cat XL program itself. For example, we might make the assumption that reinsurers will never price a layer at less than 1% ROL, no matter the underlying exposure. This information is not incorporated into f(x) but we could incorporate it into g(x).

Let n – 1 = the number of layers in the original Cat XL program. (Island Insurance has 5 layers in the original program, so n – 1 = 5). Then let g(x) be a spline with n + 1 segments (so the Island spline will have 7 segments). Let the first segment be linear, the next n – 1 segments be quadratic (representing the original layers) and the last segment be linear. Such a g(x) can be constructed in a way that resolves weakness #1 and is bounded on the top and on the bottom with a maximum ROL and a minimum ROL.

How do we do this? First, let us write down the equations for g(x). We count the n + 1 segments as 0, 1, ..., n, (n = 6 for Island Insurance). Then segments 0 and n are linear and segments 1, 2, ..., n – 1 are quadratic (these correspond to the n – 1 layers in the original Cat XL program). Let's denote the formula for segment i (or layer i) as $g_i(x)$ where $g_i(x)$ is defined on the interval (LP_i, RP_i). Then we have:

- $g_i(x) = a_i + b_i x$ for i = 0 and i = n (first and last segments are linear)
- $g_i(x) = a_i + b_i x + c_i x^2$ for i = 1, 2, ..., n – 1 (middle layers are quadratic)

Where:

- $LP_0 = 0$
- $LP_i = \frac{D_i}{TIV}$ (left endpoint)
- $RP_i = \frac{D_i + L_i}{TIV}$ (right endpoint)
- $RP_i = LP_{i+1}$ (endpoints are connected)
- D_i is the deductible of the i-th layer of the original Cat XL program (i = 1, 2, ... , n – 1)
- L_i is the limit of the i-th layer of the original Cat XL program (i = 1, 2, ... , n – 1)
- RP_n is a point beyond the original Cat XL program (maximum right endpoint)

How do we pick the coefficients a_i, b_i, c_i for g(x)? We want the following conditions to hold:

Condition #1 (Continuity): We want g(x) to be a continuous function; that is, we want the $g_i(x)$ to be connected at the endpoints. Here we have the equations $g_i(RP_i) = g_{i+1}(LP_{i+1})$ for i = 0, 1, ..., n – 1.

Condition #2 (Smoothness): We also want the first derivative $g'(x)$ to be continuous. In other words, we want the function g(x) to be smooth. This is a necessary condition for g(x) to be considered a quadratic spline. Here we have the equations $g_i^{'}(RP_i) = g_{i+1}^{'}(LP_{i+1})$ for i = 0, 1, ..., n – 1.

Condition #3 (Integration): We want for i = 1, 2, ..., n – 1 that $\int_{LP_i}^{RP_i} g_i(x)dx = p_i$ where $p_i = \frac{Cost_i}{TIV}$ and $Cost_i = ROL_i \times L_i$. $Cost_i$, ROL_i and L_i are the known Cost, ROL and Limit of the i-th layer of the original Cat XL program.

This will immediately resolve weakness #1 of the power curve as we are in essence “forcing” g(x) to integrate over the original layers to the original prices.

Condition #4 (Maximum): We want to bound g(x) on the top. We let g(0) = $g_0(0) = a_0 + b_0 \times 0 = a_0 = ROL_{MAX}$. This resolves weakness #3.

Condition #5 (Minimum): We want to bound g(x) on the bottom. We let $g(RP_n) = g_n(RP_n) = a_n + b_n \times RP_n = ROL_{MIN}$

Note that for conditions 4 and 5 the user is required to make a selection for the variables ROL_{MAX}, ROL_{MIN} and RP_n. ROL_{MAX}, the maximum possible ROL, would be the ROL charged for a theoretical layer with no deductible and infinitesimal limit. ROL_{MIN} is the lowest possible ROL, which is reached at some point RP_n which lies beyond the limit of the original Cat XL program. How do we make these selections? This is a highly judgmental step. Here are some ideas:

- We could look at Cat XL programs for companies similar to the one we are pricing (if available) and take into consideration the max and min for those programs.
- We could set RP_n as the point beyond which no coverage would ever actually be purchased.

- For ROL_{MIN} we could consider the values taken on by the power curves at RP_n (i.e. $f(RP_n)$ and $f^*(RP_n)$).
- We could simply look at the curve visually and see what selections make it the "smoothest".

We now provide all of the known variables for Island Insurance in a table, the variables that we will need to set up the equations in Conditions 1 - 5:

Table 5 - Known Variables to Solve for Island Insurance Spline						
TIV	2,700,000,000					
			LP = D / TIV	**RP = (D + L) / TIV**		**(L x R) / TIV**
Layer i	**Limit - L**	**Deductible - D**	**Left Endpt % - LP**	**Right Endpt % - RP**	**ROL - R**	**Cost % - p**
Layer 0	*5,000,000*	*0*	*0.00%*	*0.19%*	*n/a*	*n/a*
Layer 1	5,000,000	5,000,000	0.19%	0.37%	20.70%	0.038%
Layer 2	10,000,000	10,000,000	0.37%	0.74%	14.55%	0.054%
Layer 3	30,000,000	20,000,000	0.74%	1.85%	10.20%	0.113%
Layer 4	50,000,000	50,000,000	1.85%	3.70%	6.42%	0.119%
Layer 5	55,000,000	100,000,000	3.70%	5.74%	3.75%	0.076%
Layer 6	*7,000,000*	*155,000,000*	*5.74%*	*6.00%*	*n/a*	*n/a*
Total	**150,000,000**	**5,000,000**			**7.22%**	**0.401%**

Once again, for Island Insurance there are 5 layers in the original program and n = 6. We invite the reader to inspect this table in the workbook on tab #8.1. For Island Insurance, we make the following selections:

- Let RP_n = 6.00% (cell F15)
- Let ROL_{MAX} = 40.00% (cell G18)
- Let ROL_{MIN} = 3.00% (cell G19)

It may be instructive for the reader to inspect the calculation of the endpoints (columns E and F) as well as the calculation of the p_i (column H) in Excel.

6. SOLVING THE SPLINE CURVE PARAMETERS

Our goal now is to use the variables in Table 5 to set up the equations from Conditions 1-5. We will then use the system of equations to solve for the coefficients a_i, b_i, c_i and thus solve g(x).

First some notes on counting the number of equations:

- We have 3n + 1 equations. The continuity equations (from Condition 1) provide n equations (if there are n + 1 segments then there are n equations between the segments). The smoothness equations (from Condition 2) also provide n equations. The integration equations (from Condition 3) provide n – 1 equations (as there are n – 1 original layers). Finally the boundedness equations (from Conditions 4 and 5) provide 2 equations. Adding them all up we get a grand total of n + n + (n – 1) + 2 = 3n + 1 equations.

- Thus for Island Insurance (n = 6) we have 19 equations: 6 equations for the continuity between the 7 segments, 6 equations for the smoothness between the 7 segments, 5 equations so that g(x) integrate to the original Cat XL prices on each original layer, and 2 equations for the boundedness conditions.
- All of the equations are linear. Once we plug in the knowns ($LP_i, RP_i, p_i, ROL_{MAX}, ROL_{MIN}$) then the equations all reduce to linear equations with a_i, b_i, c_i as the unknowns.
- Counting up the number of unknown variables in the 3n + 1 equations we also have 3n + 1 unknowns. $g_0(x)$ and $g_n(x)$ each have 2 unknown coefficients (a_i, b_i), and each of the n – 1 $g_i(x)$ has 3 unknown coefficients (a_i, b_i, c_i), for a grand total of 2 + 2 + 3 x (n – 1) = 3n + 1 unknowns.

Thus we have a system of 3n + 1 linear equations with 3n + 1 unknown variables (the coefficients), allowing us to use matrix algebra to solve for those coefficients.

What exactly do the 19 linear equations look like for Island Insurance? We present them in the following table:

Table 6 - 19 Linear Equations for Island Insurance

Eqn #	Condition / Equation Description	General Form	Expanded Form
1	1 Continuity b/w layers 0 & 1	$g_0(RP_0) = g_1(LP_1)$	$a_0 + b_0*RP_0 = a_1 + b_1*LP_1 + c_1*LP_1^2$
2	1 Continuity b/w layers 1 & 2	$g_1(RP_1) = g_2(LP_2)$	$a_1 + b_1*RP_1 + c_1*RP_1^2 = a_2 + b_2*LP_2 + c_2*LP_2^2$
3	1 Continuity b/w layers 2 & 3	$g_2(RP_2) = g_3(LP_3)$	$a_2 + b_2*RP_2 + c_2*RP_2^2 = a_3 + b_3*LP_3 + c_3*LP_3^2$
4	1 Continuity b/w layers 3 & 4	$g_3(RP_3) = g_4(LP_4)$	$a_3 + b_3*RP_3 + c_3*RP_3^2 = a_4 + b_4*LP_4 + c_4*LP_4^2$
5	1 Continuity b/w layers 4 & 5	$g_4(RP_4) = g_5(LP_5)$	$a_4 + b_4*RP_4 + c_4*RP_4^2 = a_5 + b_5*LP_5 + c_5*LP_5^2$
6	1 Continuity b/w layers 5 & 6	$g_5(RP_5) = g_6(LP_6)$	$a_5 + b_5*RP_5 + c_5*RP_5^2 = a_6 + b_6*LP_6$
7	2 Smoothness b/w layers 0 & 1	$g_0'(RP_0) = g_1'(LP_1)$	$b_0 = b_1 + 2*c_1*LP_1$
8	2 Smoothness b/w layers 1 & 2	$g_1'(RP_1) = g_2'(LP_2)$	$b_1 + 2*c_1*RP_1 = b_2 + 2*c_2*LP_2$
9	2 Smoothness b/w layers 2 & 3	$g_2'(RP_2) = g_3'(LP_3)$	$b_2 + 2*c_2*RP_2 = b_3 + 2*c_3*LP_3$
10	2 Smoothness b/w layers 3 & 4	$g_3'(RP_3) = g_4'(LP_4)$	$b_3 + 2*c_3*RP_3 = b_4 + 2*c_4*LP_4$
11	2 Smoothness b/w layers 4 & 5	$g_4'(RP_4) = g_5'(LP_5)$	$b_4 + 2*c_4*RP_4 = b_5 + 2*c_5*LP_5$
12	2 Smoothness b/w layers 5 & 6	$g_5'(RP_5) = g_6'(LP_6)$	$b_5 + 2*c_5*RP_5 = b_6$
13	3 Area - Layer 1	$\int g_1(x) = p_1$	$a_1*(RP_1 - LP_1) + ½*b_1*(RP_1^2 - LP_1^2) + ⅓*c_1*(RP_1^3 - LP_1^3) = p_1$
14	3 Area - Layer 2	$\int g_2(x) = p_2$	$a_2*(RP_2 - LP_2) + ½*b_2*(RP_2^2 - LP_2^2) + ⅓*c_2*(RP_2^3 - LP_2^3) = p_2$
15	3 Area - Layer 3	$\int g_3(x) = p_3$	$a_3*(RP_3 - LP_3) + ½*b_3*(RP_3^2 - LP_3^2) + ⅓*c_3*(RP_3^3 - LP_3^3) = p_3$
16	3 Area - Layer 4	$\int g_4(x) = p_4$	$a_4*(RP_4 - LP_4) + ½*b_4*(RP_4^2 - LP_4^2) + ⅓*c_4*(RP_4^3 - LP_4^3) = p_4$
17	3 Area - Layer 5	$\int g_5(x) = p_5$	$a_5*(RP_5 - LP_5) + ½*b_5*(RP_5^2 - LP_5^2) + ⅓*c_5*(RP_5^3 - LP_5^3) = p_5$
18	4 Maximum ROL	$g_0(LP_0) = ROL_{MAX}$	$a_0 + b_0*LP_0 = a_0 = ROL_{MAX}$
19	5 Minimum ROL	$g_6(RP_6) = ROL_{MIN}$	$a_6 + b_6*RP_6 = ROL_{MIN}$

These 19 equations have 19 unknowns: $a_0, b_0, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, a_4, b_4, c_4, a_5, b_5, c_5, a_6, b_6$.

These equations can also be found in tab #8.2, although given that the workbook is designed to handle up to 8 original layers, there are some dummy equations listed there too (i.e. $a_7 = b_7 = c_7 = ... = a_9 = b_9 = 0$).

We then convert the 19 equations in Table 6 to the following form:
$k_{j1}a_0 + k_{j2}b_0 + k_{j3}a_1 + k_{j4}b_1 + k_{j5}c_1 + ... + k_{j18}a_6 + k_{j19}b_6 = s_j$.
That is to say, we convert each equation into a linear combination of the unknowns on the left hand side and a solution constant on the right hand side. k_{ji} is a known factor for the j-th equation and the i-th unknown variable. s_j is the solution constant to the j-th equation.

For example, let's take equation 2 which is $a_1 + b_1 RP_1 + c_1 RP_1^2 = a_2 + b_2 LP_2 + c_2 LP_2^2$. From Table 5, we have that $RP_1 = LP_2 = 0.37\%$. Then equation 2 in the prescribed format is as follows:
$a_1 + 0.37\% \times b_1 + 0.001369\% \times c_1 - a_2 - 0.37\% \times b_2 - 0.001369\% \times c_2 = 0$. Note that all the other unknowns have a factor of 0 and are not shown here.

Converting the 19 equations to matrix form, we take the k_{ji} of the left hand side and let **A** = $\begin{pmatrix} k_{1,1} & \cdots & k_{1,19} \\ \vdots & \ddots & \vdots \\ k_{19,1} & \cdots & k_{19,19} \end{pmatrix}$.

We also form the unknown equation vector **X** = $\begin{pmatrix} a_0 \\ \vdots \\ b_6 \end{pmatrix}$ and the solution vector **B** = $\begin{pmatrix} s_1 \\ \vdots \\ s_{19} \end{pmatrix}$.

Thus we have the equation **A * X = B**. Finally, we use matrix algebra to find the inverse of **A**, **A**$^{-1}$. Then **X** = **A**$^{-1}$ *** B** and we have solved for all the unknown coefficients simultaneously, thus solving g(x).
The full matrices **A**, **A**$^{-1}$ and the vectors **B** and **X** can all be found on tab #8.2.

The solved coefficients of g(x) – the spline curve for Island Insurance – are summarized in the following table:

Table 7 - Island Insurance Layer Summary and Solved Spline Coefficients

TIV	2,700,000,000								
			LP = D / TIV	**RP = (D + L) / TIV**		**(L x R) / TIV**			
Layer i	**Limit - L**	**Deductible - D**	**Left Endpt % - LP**	**Right Endpt % - RP**	**ROL - R**	**Cost % - p**	**a**	**b**	**c**
Layer 0	*5,000,000*	*0*	*0.00%*	*0.19%*	*n/a*	*n/a*	0.40	-75.83	
Layer 1	5,000,000	5,000,000	0.19%	0.37%	20.70%	0.038%	0.45	-132.95	15422.37
Layer 2	10,000,000	10,000,000	0.37%	0.74%	14.55%	0.054%	0.27	-31.82	1769.90
Layer 3	30,000,000	20,000,000	0.74%	1.85%	10.20%	0.113%	0.18	-7.77	146.95
Layer 4	50,000,000	50,000,000	1.85%	3.70%	6.42%	0.119%	0.13	-3.23	24.29
Layer 5	55,000,000	100,000,000	3.70%	5.74%	3.75%	0.076%	0.14	-3.52	28.22
Layer 6	*7,000,000*	*155,000,000*	*5.74%*	*6.00%*	*n/a*	*n/a*	*0.05*	*-0.28*	
Total	**150,000,000**	**5,000,000**			**7.22%**	**0.401%**			

These solved coefficients can be found on tab #8.1. Let's now check to see if $g(x)$ is working the way we want it to work by calculating the values of $g(x)$ and $g'(x)$ on the segment endpoints, and integrating $g(x)$ across the segments:

Table 8 - Island Insurance Layer Summary and Verification of Properties of Spline Curve											
TIV	2,700,000,000		**LP =**	**RP =**		**(L x R) /**					
		Deductible	**D / TIV**	**(D + L) / TIV**	**ROL**	**TIV**					
Layer i	**Limit - L**	**D**	**Left Endpt**	**Right Endpt**	**R**	**Cost % - p**	**g(LP)**	**g(RP)**	**g'(LP)**	**g'(RP)**	**$\int_{LP}^{RP} g(x)$**
Layer 0	*5,000,000*	*0*	*0.00%*	*0.19%*	*n/a*	*n/a*	40.00%	25.96%	-75.83	-75.83	n/a
Layer 1	5,000,000	5,000,000	0.19%	0.37%	20.70%	0.038%	25.96%	17.20%	-75.83	-18.71	0.038%
Layer 2	10,000,000	10,000,000	0.37%	0.74%	14.55%	0.054%	17.20%	12.70%	-18.71	-5.60	0.054%
Layer 3	30,000,000	20,000,000	0.74%	1.85%	10.20%	0.113%	12.70%	8.30%	-5.60	-2.33	0.113%
Layer 4	50,000,000	50,000,000	1.85%	3.70%	6.42%	0.119%	8.30%	4.82%	-2.33	-1.43	0.119%
Layer 5	55,000,000	100,000,000	3.70%	5.74%	3.75%	0.076%	4.82%	3.07%	-1.43	-0.28	0.076%
Layer 6	*7,000,000*	*155,000,000*	*5.74%*	*6.00%*	*n/a*	*n/a*	3.07%	3.00%	-0.28	-0.28	n/a
Total	**150,000,000**	**5,000,000**			**7.22%**	**0.401%**					

Checking Condition #1 (Continuity): Notice that the value that $g(x)$ takes at the right of segment i is equal to the value that $g(x)$ takes at the left of segment i + 1. For example, $g_0(RP_0) = g_1(LP_1)$ = 25.96%. This implies that $g(x)$ is continuous.

Checking Condition #2 (Smoothness): Similarly we notice that the value that $g'(x)$ takes at the right of segment i is equal to the value that $g'(x)$ takes at the left of segment i + 1. For example, $g_0'(RP_0) = g_1'(LP_1)$ = -75.83. This implies that $g'(x)$ is continuous (i.e. that $g(x)$ is smooth). Note that the derivative is negative throughout, which means that $g(x)$ is decreasing throughout. Also note that the derivative while negative is also increasing, which means that $g(x)$ is concave up (the power curves f(x) and $f^*(x)$ are also concave up).

Checking Condition #3 (Integration): Notice that the integral of each $g_i(x)$ on its defined interval LP_i to RP_i is equal to p_i (i.e. the last column in Table 8, $\int_{LP_i}^{RP_i} g(x)dx$, is equal to the 7th column in Table 8, p_i). Thus if we use g(x) to price the original program with no change in TIV, we get the same ROLs as the original program.

Checking Conditions #4 and #5 (Maximum and Minimum): Notice that $g_0(LP_0)$ = g(0%) = 40.00% = ROL_{MAX} and $g_6(RP_6)$ = g(6.00%) = 3.00% = ROL_{MIN}.

These verifications can also be found on tab #8.1.

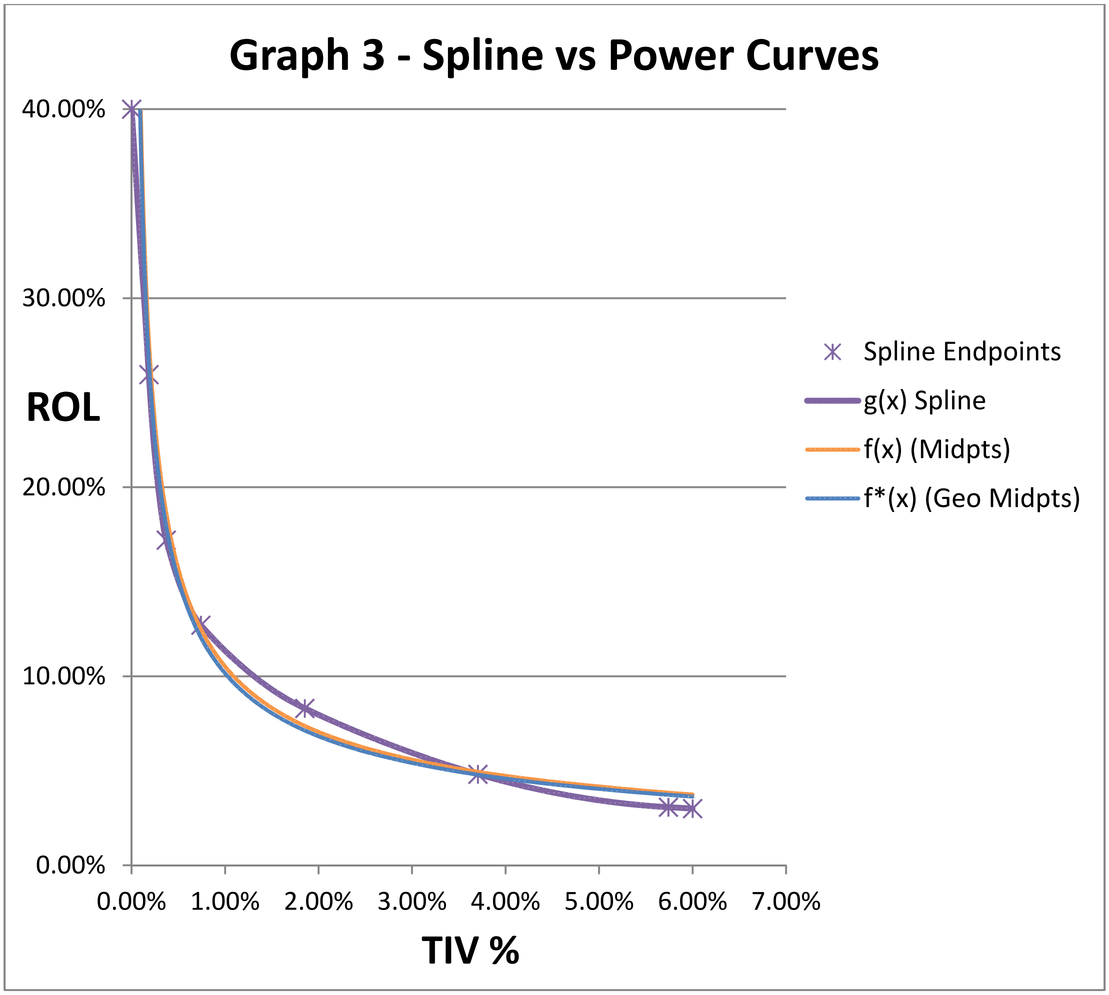
7. COMPARISON OF THE METHODS

We have now solved for f(x), $f^*(x)$ and g(x). Let's use all three of them to price out the new Cat XL program for Island Insurance with the new TIV:

Table 9 - Pricing of New Program using all the Methods

New TIV 3,000,000,000			Endpoints and Midpoints				Power Curve f(x)		Power Curve $f^*(x)$		Spline Method	
New Program Layering												
		Deductible					f(MP)	L x ROL_1	$f^*(GMP)$	L x ROL_2	$Cost_3$ / L	$\int_{LP}^{RP} g(x)$ * TIV
Layer i	Limit - L	D	LP	GMP	MP	RP	ROL_1	$Cost_1$	ROL_2	$Cost_2$	ROL_3	$Cost_3$
Layer 1	7,500,000	7,500,000	0.25%	0.35%	0.38%	0.50%	18.51%	1,388,155	18.42%	1,381,650	17.53%	1,314,627
Layer 2	20,000,000	15,000,000	0.50%	0.76%	0.83%	1.17%	11.69%	2,337,163	11.85%	2,370,376	12.37%	2,473,283
Layer 3	50,000,000	35,000,000	1.17%	1.82%	2.00%	2.83%	7.06%	3,529,088	7.21%	3,606,327	8.10%	4,047,793
Layer 4	90,000,000	85,000,000	2.83%	4.07%	4.33%	5.83%	4.52%	4,069,582	4.55%	4,094,577	4.24%	3,813,139
Total	**167,500,000**	**7,500,000**					**6.76%**	**11,323,987**	**6.84%**	**11,452,929**	**6.95%**	**11,648,842**

This table can be found on tab #10. We see that using geometric midpoints, i.e. $f^*(x)$, results in ROLs that are generally higher than using arithmetic midpoints, i.e. f(x). We also see that, for this example, using the spline method results in the highest overall ROL (6.95%), with significant variation from layer to layer. For example, the cost of layer 3 is 14.7% higher under g(x) than under f(x) (8.10% / 7.06%), yet the cost of layer 4 is 6.2% lower (4.24% / 4.52%). This variation from layer to layer is due to parameterized flexibility of the spline curve, which becomes more apparent when we observe it visually:



The spline endpoints marked by asterisks indicate the left and right endpoints of the original Cat XL program, in addition to LP_0 and RP_6. As can be seen, the spline is strictly decreasing throughout and is higher than the power curves on original layers 3 and 4 and lower on original layer 5. We can also clearly see that g(x) is bounded at the top, that g(0) = 40.00%.

We encourage the reader to try pricing out a new Cat XL program in the Excel workbook by entering in the new Cat XL structure in green in tab #4 and then examining the output pricing in tab #10. Here are some structures to test:

- Enter in the original TIV = $2,700,000,000 and the original structure: $5m xs $5m, $10m xs $10m, $30m xs $20m, $50m xs $50m and $55m xs $100m. Compare tab #1 with tab #10. Notice that the layer ROLs are preserved under the spline curve g(x) but are not preserved under the power curves. The correct overall ROL for the program is 7.22%.
- Now go back to tab #4, keep the TIV as it is, and enter the following three layers: $20m xs $5m, $30m xs $25m and $100m xs $55m. This gives an overall program of $150m xs $5m, the same as before. Notice in tab #10 that the overall ROL under the spline method stays the same at 7.22%. This is a property of using the spline method – while individual layer prices will be different, the overall price will be the same because integration is additive: $\int_{LP_1}^{RP_1} g(x)dx + \int_{LP_2}^{RP_2} g(x)dx = \int_{LP_1}^{RP_2} g(x)dx$.

8. ADVANTAGES OF POWER CURVES

While we have presented the spline method as superior to power curves in that it resolves the three key weaknesses in section 4, there are ways in which power curves are superior to splines:

- Power curves are easier to set up and explain than splines.
- Splines require ROL_{MAX} and ROL_{MIN} to be specified judgmentally in the model; the power curves require no such selections.
- The power curve function is $f(x) = a * x^{-b}$ and its derivative is $f'(x) = -a * b * x^{-b-1}$. Since the derivative is negative for all x, the power curve is strictly decreasing. But for spline curves there is nothing in the definition that enforces this property. Upon visual inspection we may occasionally find a spline curve with a region that is not strictly decreasing. In those cases we might be able to "fix" the curve by selecting a different ROL_{MAX} and ROL_{MIN}.
- Finally, an application: let's say we have the Cat XL programs of several similar insurance companies (e.g. competitors of Island Insurance who also write property policies exclusively on the island). We can then fit a power curve through the midpoints of ALL the layers of ALL the Cat XL programs, thus creating a consolidated market curve for the entire island, not just Island Insurance. It is not obvious how to create such a consolidated market curve using a spline.

9. SUMMARY

In this paper we have presented the concept of pricing a catastrophe excess of loss program (Cat XL) using a market curve. Pricing with such a market curve is simple in that it only requires the total insured value (TIV) of the new program to be priced, and a benchmark program (such as last year's Cat XL program), and does not require the use of catastrophe modelling output.
We then presented the simplest market curve, which is the power curve. The power curve fits a function of the form $f(x) = a * x^{-b}$ to the midpoints (arithmetic or geometric) of the benchmark program, and then this curve is used to price out the new program. We showed, however, that the power curve has three key weaknesses.

We then proposed a new market curve, a spline function, and the use of integration instead of taking the midpoints of layers, which resolves the three key weaknesses of the power curve. We showed how to solve for the spline, which involves solving a system of linear equations.

We provided an Excel workbook that allows the reader to test all the methods.

10. FURTHER RESEARCH

The power curve function takes the form $f(x) = a * x^{-b}$, however other variations could be investigated:

- **Power Curve with a constant:** $f(x) = a * x^{-b} + c$ (Notice that ROL_{MIN} = c)
- **Exponential Decay:** $f(x) = a * b^{-x}$ (Notice that ROL_{MAX} = a)
- **Exponential Decay with a constant:** $f(x) = a * b^{-x} + c$ (Notice that ROL_{MAX} = a + c and ROL_{MIN} = c)

In addition, we could investigate some of the simplifying assumptions that we made in the paper:

- The issue of reinstatements could be studied. What happens if different layers have different reinstatement conditions?
- What happens if we assume a known reinsurance market cycle as opposed to "flat", unchanging rates?

Finally, we could try to develop a formula to relate the market curve and market pricing to the underlying catastrophe exposure.

11. ACKNOWLEDGMENTS

The author would like to thank Neil Franklin for teaching him the original power curve method, Kirk Conrad for his guidance and encouragement, and Neil Bodoff for reading the first draft and providing feedback.

12. REFERENCES

As stated earlier in the paper, the author is not aware of any published document that presents the power curve method, despite the fact that it has been used in the reinsurance market since the early 1990s. However, we present two references for the general edification of the reader. The first is the Clark paper "Basics of Reinsurance Pricing" and the second is the "Loss Models" textbook, which provides a background on spline curves (more advanced splines).

[1] D. Clark, "Basics of Reinsurance Pricing," CAS Study Note, 1996.

[2] Klugman, Stuart A., Harry H. Panjer and G.E. Willmot, G.E. *Loss Models: from Data to Decisions*. Wiley Series in Probability and Statistics, 1998.