The Cost of Financing Insurance

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by

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Abstract

This paper uses Dynamic Financial Analysis (DFA), to attack one of the longest-running problems in actuarial science — that of determining the appropriate profit loading for a line of insurance. For an insurance company, the cost of financing insurance is its (dollar) cost of capital plus the net cost of its reinsurance. The profit loading for a line of insurance is the cost of financing allocated to the line of insurance. Important considerations in determining this allocation include: (1) how much does the line contribute to the need for capital; and (2) how long must the insurer hold capital to support the uncertainty in its underwriting results.

Introduction

This paper uses the recently coined actuarial discipline, Dynamic Financial Analysis (DFA), to attack one of the longest-running problems in actuarial science – that of determining the appropriate profit loading for a line of insurance. Susan Szkoda [8], in her five-part article beginning in the May 1997 *Actuarial Review*, defines DFA as "a process for analyzing the financial condition of an insurance entity. Financial condition refers to the ability of the entity's capital and surplus to adequately support future operations through a currently unknown future environment. ... In a very real sense, DFA requires the actuary to evolve into a financial risk manager."

In this paper, I will attempt to derive a logically consistent method for using DFA to determine the profit loading on a line of insurance. I will then apply the method to one hypothetical insurer.

The ABC Insurance Company is a multiline insurance company. Its goal is to obtain an above-average return on equity by setting profitability targets for each of its underwriting divisions that reflect the cost of capital needed to support each division's contribution to the overall underwriting risk. If ABC expects an underwriting division's long-term results to fall below its target, the company intends to get out of that line of insurance.

ABC's management wants to use the following considerations as input into its decisions.

- How much capital must the company hold? While ABC's management recognizes the important role of regulators and rating agencies in determining an insurer's capital, the managers feel that controlling the insurer's risk, as measured by its statistical distribution of outcomes, provides a meaningful yardstick for setting profitability targets.

- How long must the company hold capital? The company may not know its underwriting results of its liability lines of insurance for several years. As long as there is uncertainty in the final result, the company must hold some capital. The profitability targets for each line of insurance should reflect the cost of holding capital until all claims are settled.

- How much investment income does the insurance operation generate? As the insurer is holding capital for the contingency of unusually high losses, it is also earning investment income on its capital. The profitability targets for each line of insurance should also reflect the investment earnings generated by each line of business.

- How closely correlated are the losses in the various lines of insurance? The textbook illustrations of the economic value of insurance often assume that insured accidents are independent events. Positive correlation increases the amount of capital needed and hence its cost. The profitability targets for each line of insurance should reflect this cost

- What is the effect of reinsurance? In place of raising capital, an insurer may rely on reinsurance to provide security for its ability to pay losses. The effect of reinsurance is to replace part of the cost of capital with the cost of reinsurance. The profitability targets should reflect both the cost and benefit of reinsurance for each line of insurance.

I define the cost of financing an insurance company as the combined cost of capital and the net cost of reinsurance (that is, the premium less the expected reinsurance recovery). The ABC Insurance Company wants to allocate its cost of financing back to its individual underwriting divisions.

ABC will add the allocated cost of financing insurance to the expected losses and the other allocated expenses to obtain target combined ratios for each underwriting division in the company.

2. Outline

The final product of this analysis will be a table of target combined ratios for underwriting divisions of the ABC Insurance Company. As we move toward that end, I will cover a number of actuarial and financial concepts. Here are the highlights of our trip.

- Section 3 discusses the concept of capital and the insurer's aggregate loss distribution. The typical insurer writes several lines of insurance and so we must get the distribution of the sum of the random losses from each line. That means we must consider the possibility that the losses in each line are correlated.

- Section 4 introduces the concept of measures of risk. The section begins with four axioms that risk measures should satisfy. Next I state a theorem that characterizes all risk measures that satisfy these axioms. I then discuss how well some of the popular actuarial risk measures fit into this axiomatic framework.

- Section 5 discusses the cost of capital. We express the amount of needed capital in terms of the insurer's chosen measure of risk. The insurance company's investors provide this capital — at a cost. The policyholder must ultimately pay the cost of providing this capital. This section gives a formula to allocate the cost of capital to the various underwriting divisions — which in turn must decide how to allocate their allocated cost of capital to their individual policyholders.

- Section 6 discusses the effect of long-tailed lines of insurance. An insurer does not know the underwriting result for the typical liability line for insurance several years. As long as there is uncertainty in the final result, the insurer must hold some capital. This capital has a cost. This section shows how to allocate the cost back to the appropriate underwriting division.

- Section 7 discusses reinsurance. In place of raising capital, an insurer may rely on reinsurance to provide security for its ability to pay losses. The effect of reinsurance is to replace the cost of capital with the net cost of reinsurance. Introducing reinsurance forces us to move from the very specific concept of the cost of capital to the more general concept of the cost of financing insurance.

- Sections 8 and 9 put all the pieces together to calculate the cost of financing insurance for each underwriting division. We will calculate the cost of financing with and without reinsurance, and for two different measures of risk.

- Section 10 translates the results into target combined ratios.
- Section 11 finishes the paper with some concluding remarks.

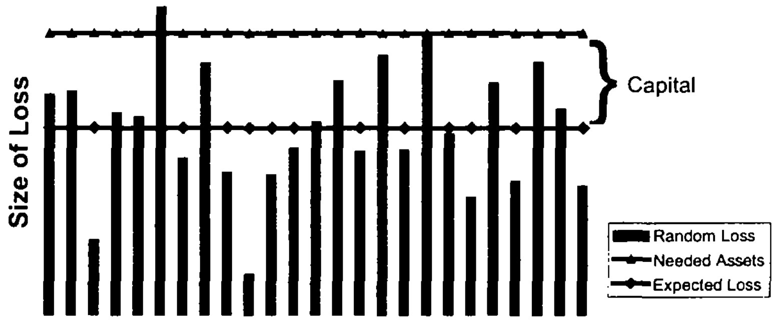
I am writing this paper to provide a conceptual overview of how to apply DFA to the management of underwriting risk. A comprehensive DFA analysis on a real insurance company involves a myriad of details that, if considered here, would make the underlying concepts harder to grasp. Therefore, I have made a number of simplifications, the most important of which is the model of the insurer's losses.

3. Capital and the Distribution of an Insurer's Aggregate Losses

The first step in our analysis will be to determine how much capital an insurer needs to be "reasonably" certain that it can pay its claims. Often, the insurer will be able to pay its claims from the expected loss portion of its premium income. But in some years losses are above average, and the insurer needs additional capital to make good its pledge to its insureds. Although the insureds would like to be absolutely certain that the insurer has enough capital to pay its claims, in practice, insureds are willing to allow for the "rare" possibility that the insurer will have insufficient funds. Chart 3.1 illustrates the idea.

We will further refine our notion of "rare" in the next section.

Chart 3.1



The total assets needed to cover losses is equal to the sum of:
(1) the expected loss, which comes from the premium, plus
(2) the capital which comes from the insurer's investors.

We need to consider the insurer's distribution of aggregate losses when determining the amount of capital needed. The most common description of an insurer's aggregate losses is the collective risk model. That model describes the insurer's losses in terms of a random claim count and a random claim size for each line of insurance. The model allows us to account for several features of the insurer's business including inflation, deductibles, policy limits, and reinsurance.

Conceptually, the easiest way to implement the collective risk model is to perform a Monte Carlo simulation. There are practical problems in doing this because the simulations can take a considerable amount of time. If the insurer wants to consider a number of alternative strategies that involve purchasing reinsurance and/or modifying its book of business, the time needed to do the computations can limit the number of alternatives the company can consider. There are faster ways to perform collective risk model calculations, but those methods rely on advanced mathematical techniques.

In writing this paper, I have chosen to move most of the problems to the background by building a simplified aggregate loss model. The model consists of four lines of insurance. We will describe the aggregate loss distribution for each line of insurance by a normal distribution with mean μ and standard deviation σ. The lines will have varying risk and loss payment characteristics. There will be an additional catastrophe loss that occurs with a low probability. With that simplified model, we can perform the necessary convolutions to sum the random losses and instantaneously calculate the various statistics needed to do the financial analysis.

The example we will follow throughout this paper will be the ABC Insurance Company. For the accident year 2002, it expects to pay $250 million in losses. For prior accident years it holds reserves totaling $227 million. The following table presents the outstanding liabilities for each line of insurance.

Table 3.1
By Line Loss Statistics for ABC Insurance Company

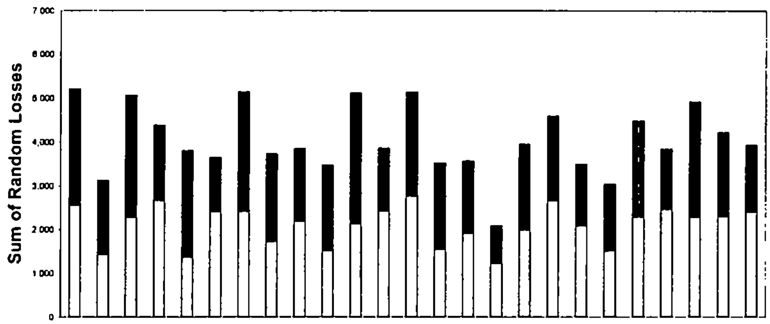
Outstanding Loss + ALAE Parameters

Line & AY	μ	σ/μ	σ
GL-1998	2,000,000	0.270	540,000
GL-1999	10,000,000	0.180	1,800,000
GL-2000	25,000,000	0.120	3,000,000
GL-2001	45,000,000	0.090	4,050,000
GL-2002	70,000,000	0.060	4,200,000
PL-1998	5,000,000	0.300	1,500,000
PL-1999	15,000,000	0.200	3,000,000
PL-2000	30,000,000	0.150	4,500,000
PL-2001	50,000,000	0.100	5,000,000
PL-2002	70,000,000	0.080	5,600,000
Auto-2000	10,000,000	0.140	1,400,000
Auto-2001	35,000,000	0.080	2,800,000
Auto-2002	70,000,000	0.050	3,500,000
Prop-2002	35,000,000	0.090	3,150,000
Cat-2002	5,000,000	7.000	35,000,000

The catastrophe loss distribution consists of a loss of $250 million with probability 0.02, and a loss of zero with probability 0.98.

An important consideration when analyzing aggregate loss distributions is correlation. Consider an example with independent random losses X_1 and X_2, each with mean 2000 and standard deviation 500. Chart 3.2 shows a plot of the sum of X_1 and X_2 for 25 random selections.

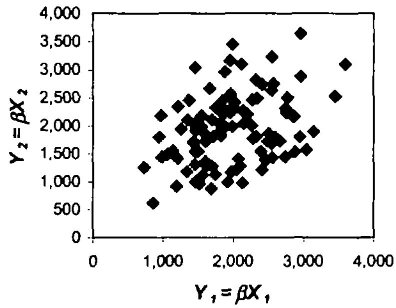
Chart 3.2
Uncorrelated Losses



An insurer covering X_1+X_2 would need assets slightly over \$5,000 to cover the losses shown.

Now, let's complicate the example by first taking a random multiplier, β, of 0.7, 1.0, or 1.3. (The corresponding probabilities of β are 1/6, 2/3, and 1/6 respectively.) Next we take X_1 and X_2 as above and then set $Y_1 = \beta X_1$ and $Y_2 = \beta X_2$. Chart 3.3 shows a plot of 100 randomly selected pairs (Y_1, Y_2). As Chart 3.3 illustrates, Y_1 and Y_2 are correlated.

Chart 3.3

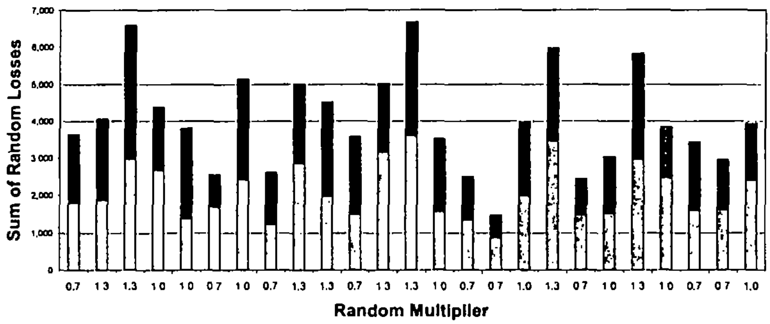


Large values of Y_1 (=βX_1) are statistically associated with large values of Y_2 (=βX_2). Hence Y_1 and Y_2 are positively correlated.

Adding a pair of correlated random losses produces a more volatile distribution than adding a pair of similar but uncorrelated random losses. Chart 3.4 shows the effect of adding the Y's corresponding to the X's in Chart 3.2.

As Charts 3.2 and 3.4 clearly illustrate, an insurer would need more assets to cover Y_1+Y_2 than it would need to cover X_1+X_2. Now since E[β] =1, we have that E[Y_1+Y_2] = E[X_1+X_2]. Hence the insurer would need to get more capital from its investors to cover Y_1+Y_2 than it would need to cover X_1+X_2.

Chart 3.4
Correlated Losses



An insurer covering Y_1+Y_2 would need assets well over \$6,000 to cover the losses shown. That is noticeably higher than the assets needed to cover the losses X_1+X_2 shown in Chart 3.2.

Now let's apply this random multiplier idea to our model of the noncatastrophe losses of the ABC Insurance Company. For a given $b > 0$, choose random multipliers:

$$\beta = 1 - \sqrt{3b} \text{ with probability } 1/6$$
$$\beta = 1 \text{ with probability } 2/3$$
$$\beta = 1 + \sqrt{3b} \text{ with probability } 1/6$$

We have that $E[\beta] = 1$ and $Var[\beta] = b$.

We will apply the random multiplier, β, to all of ABC's noncatastrophe losses. Setting $b = 0$ forces ABC's non-catastrophe losses to be independent. Increasing b results in a greater volatility of ABC's total noncatastrophe losses. Table 3.2 gives some aggregate statistics for ABC's noncatastrophe losses over a range of b's.

Table 3.2
Aggregate Statistics for ABC's Noncatastrophe Losses

b	Standard Deviation	99th Percentile
0.00	12,899,868	502,009,504
0.01	48,948,040	577,282,947
0.02	68,010,402	612,585,449
0.03	82,794,437	639,672,796

4. Measures of Risk

4.1 Introduction

The discussion in the previous section of the assets needed to covér an insurer's potential losses has two implicit assumptions:

1. The amount of needed capital increases with the volatility of the insurer's losses.
2. It is unreasonable to require an amount of capital sufficient to cover all potential losses.

In this section, we discuss some rules for determining how much assets and capital an insurer needs to cover its losses. These rules will depend on the insurer's aggregate loss distribution. Other valid considerations, such as the quality and reputation of the insurer's management, are beyond the scope of this paper.

Most of the ideas in this section come from the paper "Coherent Measures of Risk" by Philippe Artzner, Freddy Delbaen, Jean-Marc Eber and David Heath [2]. Their paper considers the problem of setting margin requirements on an organized exchange. This problem is similar to that of setting capital requirements for insurance companies.

This paper was written for an academic audience with extensive training in probability theory. Some actuaries will have some difficulty digesting the paper itself. In this section, I will attempt to describe the paper's ideas in language that is familiar to most actuaries.

Artzner [3] has written another paper on the subject that casualty actuaries might find more accessible.

4.2 A Motivation for the Definition of Coherence

Consider the following set of ten scenarios, each with associated losses X_1, X_2, X_3 and X_4.

Table 4.1

Scenario	X_1	X_2	X_1+X_2	$X_3 = 2*X_1$	$X_4 = X_1+1$
1	1.00	0.00	1.00	2.00	2.00
2	2.00	0.00	2.00	4.00	3.00
3	3.00	0.00	3.00	6.00	4.00
4	4.00	1.00	5.00	8.00	5.00
5	3.00	2.00	5.00	6.00	4.00
6	2.00	3.00	5.00	4.00	3.00
7	1.00	4.00	5.00	2.00	2.00
8	0.00	3.00	3.00	0.00	1.00
9	0.00	2.00	2.00	0.00	1.00
10	0.00	1.00	1.00	0.00	1.00
Maximum Loss	4.00	4.00	5.00	8.00	5.00

We can think of the X_i's as random variables representing the losses of the i^{th} risk. In our examples, we shall assume that each scenario is equally likely. Let us define a measure of risk for X_i as

$$\rho(X_i) = Maximum(X_i),$$

where the maximum is taken over all ten scenarios.

That measure of risk fulfills the needs of an insurance regulator who wishes to require that the insurer have sufficient assets, quantified by $\rho(X)$, to cover the losses incurred in each of the scenarios. Premiums paid by the insureds may supply some of the assets. The remainder of the assets must be supplied as insurer capital.

Using Table 4.1 as an aid, the reader should be able to verify that the measure of risk, ρ, satisfies the following axioms.

1. Subadditivity – For all random losses X and Y,

$$\rho(X+Y) \le \rho(X)+\rho(Y).$$

2. Monotonicity – If $X \le Y$ for each scenario, then,

$$\rho(X) \le \rho(Y).$$

3. Positive Homogeneity – For all $\lambda \ge 0$ and random losses X,

$$\rho(\lambda X) = \lambda\rho(X).$$

4. Translation Invariance – For all random losses X and constants α,

$$\rho(X+\alpha) = \rho(X)+\alpha.$$

A measure of risk that satisfies these four axioms is called a **coherent measure of risk**.

The axioms describe what appear to be reasonable properties for a measure of risk.

- Subadditivity reflects the diversification of portfolios or that "a merger does not create extra risk" [5, page 5] and [2, page 5]. This is a natural requirement consistent with the role of insurance. In general, we expect mergers to reduce the risk.

- Monotonicity means that if $X < Y$ for every scenario, the assets needed to support X are less than the assets needed to support Y.

- Positive homogeneity is a limiting case of subadditivity, representing what happens when there is precisely no diversification effect [5, p. 4].

The "Standard Deviation" criterion sets the measure as the expected value of the loss plus a predetermined multiple of the standard deviation. For the scenarios listed in Table 4.3 below we have:

$$X_1 \leq X_2$$
$$\rho(X_1) \equiv E[X_1] + 2 \cdot StDev[X_1] = 5.83$$
$$\rho(X_2) \equiv E[X_2] + 2 \cdot StDev[X_2] = 5.00$$

As this example shows, the Standard Deviation criterion violates the monotonicity axiom.

Table 4.3

Scenario	X_1	X_2
1	1.00	5.00
2	2.00	5.00
3	3.00	5.00
4	4.00	5.00
5	5.00	5.00
6	5.00	5.00
7	4.00	5.00
8	3.00	5.00
9	2.00	5.00
10	1.00	5.00
$E[Loss]$	3.00	5.00
$StDev[Loss]$	1.41	0.00
$E[Loss]+2*StDev[Loss]$	5.83	5.00

Note the following.

Proposition 4.1

The Standard Deviation criterion is subadditive.

Proof

Let $\sigma_X^2 = \text{Var}[X]$, $\sigma_Y^2 = Var[Y]$, $\sigma_{X+Y}^2 = Var[X+Y]$ and $r_{XY} = Corr[X,Y]$. Then:

$$\begin{aligned} E[X+Y] + T\sigma_{X+Y} &= E[X+Y] + T\sqrt{\sigma_X^2 + 2r_{XY}\sigma_X\sigma_Y + \sigma_Y^2} \\ &\leq E[X+Y] + T\sqrt{\sigma_X^2 + 2\sigma_X\sigma_Y + \sigma_Y^2} \\ &= E[X+Y] + T\sqrt{(\sigma_X + \sigma_Y)^2} \\ &= E[X] + T\sigma_X + E[Y] + T\sigma_Y. \end{aligned}$$

■

4.3 Other Measures of Risk

It turns out that many common measures of risk used by actuaries are not coherent. Consider the following examples.

Define the "Value at Risk" or *VaR* as the smallest loss greater than a predetermined percentile of the loss distribution. This measure is similar to "Probability of Ruin" measures that actuaries have long discussed.

If our measure of risk, $\rho(X)$, is the 85th percentile of the random loss X, we have for the scenarios listed in Table 4.2 below:

$$0=\rho(X_1)+\rho(X_2)<\rho(X_1+X_2)=1.$$

As this example shows, the Value at Risk criterion violates the subadditivity axiom.

Table 4.2

Scenario	X_1	X_2	X_1+X_2
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.00
4	0.00	0.00	0.00
5	0.00	0.00	0.00
6	0.00	0.00	0.00
7	0.00	0.00	0.00
8	0.00	0.00	0.00
9	0.00	1.00	1.00
10	1.00	0.00	1.00
VaR@85%	0.00	0.00	1.00

So far, I have demonstrated that two popular statistical measures on solvency standards are not coherent. Let me now turn to a more general description of coherent measures of risk.

4.4 The Representation Theorem

Let Ω denote a finite set of scenarios. Let X be the loss incurred by the insurer under a particular business plan. We associate each loss with an element of Ω.

The representation theorem [2, Proposition 4.1, and 5, Proposition 2.1], stated here without proof, says that a measure of risk, ρ, is coherent *if and only if* there exists a family, $\mathcal{P}$, of probability measures defined on Ω such that

$$\rho(X) = \sup\{E_{\mathbf{P}}[X] | \mathbb{P} \in \mathcal{P}\}. \tag{4.1}$$

One way to construct a family of probability measures on Ω is to take a collection $\mathcal{A} = \{A_i\}_{i=1}^{m}$ of subsets of Ω with the property that $\bigcup_{i=1}^{m} A_i = \Omega$. Let n_i be the number of elements in A_i. Assume that all elements in Ω are equally likely. We then define the probability measure, $\mathbf{P}_i$, on the elements $\omega \in \Omega$ as the conditional probability, given that the element is in the set A_i, and 0 otherwise. That is:

$$\mathbb{P}_i(\omega) = \begin{cases} \dfrac{1}{n_i} & \text{if } \omega \in A_i \\ 0 & \text{if } \omega \notin A_i \end{cases}.$$

The authors [3, p. 16] refer to the collection of probability measures, $\mathcal{P}$, on the set of scenarios as "generalized scenarios."

Let's look at an example. The following table gives a set of scenarios and associated losses.

Table 4.4

Scenario	X
1	0
2	2
3	2
4	6

Let $A_1 = \{1,2\}$ and $A_2 = \{3,4\}$. We then calculate the expected values

$$E_{P_1}[X] = 1 \text{ and } E_{P_2}[X] = 4.$$

The associated coherent measure of risk, $\rho_A(X)$, is then given by

$$\rho_A(X) = \sup\{E_{P_i}[X] \,|\, i = 1,2\} = 4.$$

We can similarly construct a second coherent measure of risk, $\rho_B(X)$, on the scenarios in Table 4.4 with the subsets $B_i = \{i\}$. In that case we have $\rho_B(X) = 6$.

We can impose varying degrees of conservatism on coherent measures of risk by varying the choice of generalized scenarios.

4.5 A Proposal for a Measure of risk

The paper by Artzner *et. al.* finishes with a proposal for a measure of risk that actuaries should find easy to implement. Let's start with the formal definition of the Value at Risk (VaR). Let α be a selected probability (for example, 99%). Then

$$VaR_\alpha(X) = \inf\{x \mid \Pr\{X \le x\} > \alpha\}$$

As demonstrated in section 4.3, VaR is not a coherent measure of risk.

We now define the proposed measure in terms of the VaR. We call this measure the Tail Conditional Expectation (TCE) or Tail Value at Risk ($TVaR$).

$$TCE_\alpha(X) \equiv TVaR_\alpha(X) \equiv E\left[X \mid X \ge VaR_\alpha(X)\right)$$

The $TVaR$ is linked to a well-known criterion in recent CAS literature for solvency — the Expected Policyholder Deficit (EPD). See, for example, [1]. $EPD(t)$ is defined as the expected loss over a predetermined threshold t. It turns out that

$$TailVaR_\alpha(X) = VaR_\alpha(X) + \frac{EPD\left(VaR_\alpha(X)\right)}{1-\alpha}.$$

I will now demonstrate that the $TVaR$ is coherent under some common conditions. For any subset A of Ω, let n_A be the number of elements in A. Define the probability measure

$$\mathbb{P}_A(\omega) = \begin{cases} \frac{1}{n_A} & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}.$$

Proposition 4.2

If each element of Ω is equally likely, then the *TVaR* is a coherent measure of risk.

Proof

Let n be the number of elements in Ω. Denote the various values of X by $x_1 \leq x_2 \leq \ldots \leq x_n$. Let k be the integer with $0 \leq k < n$ such that $\alpha \in \left[\frac{k}{n}, \frac{k+1}{n}\right.$.

Since $\Pr\{X \leq x_{k+1}\} \geq \frac{k+1}{n} > \alpha$ and $\Pr\{X < x_{k+1}\} \leq \frac{k}{n} \leq \alpha$ we have that $VaR_\alpha(X) = x_{k+1}$.

Let $\mathcal{A}$ be the family of subsets of Ω with exactly $n-k$ elements. Define the family of measures $\mathcal{P} = \{\mathbb{P}_A\}_{A \in \mathcal{A}}$. By Equation 4.1, $\rho(X) = \sup\{E_{\mathbf{P}_A}[X] | A \in \mathcal{A}\}$ is a coherent measure of risk.

For any scenario, $\omega \in A$, $\Pr\{W = \omega | \omega \in A\} = \frac{1}{n-k}$.

Let *AMax* be the member of $\mathcal{A}$ with the $n-k$ largest elements; i.e., $\{x_{k+1}, x_{k+2}, \ldots, x_n\}$.

Then

$$
\begin{aligned}
TVaR_\alpha(X) &= \mathrm{E}\left[X | X \geq VaR_\alpha(X)\right] \\
&= \frac{x_{k+1} + x_{k+2} + \ldots + x_n}{n-k} \\
&= E_{\mathbf{P}_{AMax}}[X].
\end{aligned}
$$

For any other set $A \in \mathcal{A}$, $\mathrm{E}_{\mathbf{P}_A}[X] \leq \mathrm{E}_{\mathbf{P}_{AMax}}[X]$.

Thus $TVaR_\alpha(X) = \sup\{\mathrm{E}_{\mathbf{P}_A}[X] | A \in \mathcal{A}\}$ and the measure of risk is coherent.

■

In the examples below, we will use the Tail Value at Risk as our measure of risk. We will also show the results for the Standard Deviation measure. The Standard Deviation measure satisfies three of the four coherence axioms. It has the added advantage of being computationally faster. Also, the Tail Value at Risk is a new measure of risk. In my experience, whenever one proposes a new actuarial technique, there are always those who want to compare the new with the old – regardless of the justification for the change.

The Tail Value at Risk does address a common complaint that many actuaries have made about the Standard Deviation measure. The complaint is that the Standard Deviation measure penalizes the potential for unusually good results — up-side risk — as well as the potential for unusually bad results — down-side risk. The Tail Value at Risk is sensitive only to down-side risk.

5 The Cost of Capital

We will use a measure of risk, $\rho(X)$, to determine the assets needed to cover the random insured loss, X. Of that amount, $\rho(X)$, the insured's premium supplies the expected value, $E(X)$. The remainder, $C(X) \equiv \rho(X) - E(X)$, must come from the investors in the insurance company. We call $C(X)$ the insurer's capital. The insurer places that capital at risk for the purpose of covering losses in excess of $E(X)$.

For the examples in this paper, we will use $\rho(X) = TVar_\alpha(X)$ with $\alpha = 99\%$. Another insurer might set its capital by using a 99.5% $TVaR$ level or set it equal to 2 times its aggregate standard deviation. The insurer's policyholders might demand different standards for those insurers. While such standards are rarely so explicit in the real world, the rating agencies clearly have a more subjective version of this kind of standard in mind. Note the names they give to their ratings. For example, we have the "Best's Capital Adequacy Rating" and the Standard and Poor's rating of "Claims Paying Ability."

In return for placing their capital at risk, investors seek a target (that is, expected) rate of return at least as high as other investments of comparable risk. Exactly how high that rate of return is can be a matter of considerable debate. We could appeal to a financial theory such as the Capital Asset Pricing Model (CAPM). The CAPM tells us that investors will demand a higher return if the insurer's financial results move with the stock market. For example, a property insurer whose principal exposure is to natural hazards might find that its returns are independent of the market. A casualty insurer whose principal exposure is in long-tailed lines, such as workers compensation, may find that its returns are highly correlated with other segments of the stock market.

The insights of financial theory, while having an attractive rationale, have been difficult to quantify. An insurer might incorporate those insights into a target rate of return by selecting a peer group of insurers that the company expects to have similar returns and are comparably rated by the rating agencies. Such an analysis would subject these insights to the reality test of a benchmark.

Rightly or wrongly, setting a target rate of return is a routine exercise done by insurer boards of directors.

For the examples in this paper, we will use a target rate of return, denoted by e, equal to 12%.

Ultimately, the policyholders must bear the cost of providing necessary capital through the premiums they pay for the insurance. The insurer now faces the question of allocating that cost back to a diverse set of policyholders. One way of doing this is to allocate the capital to groups of policyholders (called underwriting divisions) within the company and compare their expected (dollar) return to their allocated capital. Each underwriting division then has the responsibility of obtaining the insurer's overall rate of return on its allocated capital. The underwriting division strives to earn that target through its underwriting and pricing activities.

In allocating capital to an underwriting division, we should convince ourselves that the resulting decisions implied by our allocation method make economic sense. By making "economic sense" we mean that insuring the policies in a given underwriting division does not decrease the insurer's expected rate of return.

Some mathematics will make this argument clearer. Let:

$C \equiv C(X) =$ Capital needed to support X.

$X_k =$ Random loss for the k^{th} underwriting division.

$\Delta C_k \equiv C(X) - C(X - X_k) =$ Marginal capital for the k^{th} underwriting division.

$\Delta P_k =$ Expected profit for the k^{th} underwriting division.

$P = \sum_k \Delta P_k$.

Proposition 5.1

Including the insurance policies in underwriting division k increases the overall expected rate of return if and only if underwriting division k's expected rate of return on its marginal capital is greater than the insurer's overall rate of return.

Proof

$$\frac{P - \Delta P_k}{C - \Delta C_k} < \frac{P}{C} \Leftrightarrow P \cdot \Delta C_k < C \cdot \Delta P_k \Leftrightarrow \frac{P}{C} < \frac{\Delta P_k}{\Delta C_k}$$

■

Proposition 5.1 places a lower bound on an underwriting division's expected profit for it to be economically viable with its insurance company. One might expect that it is all right to set a profitability target so that each underwriting division's expected return on its marginal capital is equal to the insurer's overall return on capital. But alas, life is not so simple. Consider the following proposition.

Proposition 5.2

Let an insurer's capital, C, be determined by a subadditive measure of risk, ρ. Then:

$\underset{k}{}\Delta C_k \leq C$ that is, the sum of the marginal capitals is less than or equal to the original capital.

Proof

I first offer the proof when there are two underwriting divisions.

$$
\begin{aligned}
&\Delta C_1 + \Delta C_2 \\
&= \rho(X_1 + X_2) - E(X_1 + X_2) - (\rho(X_2) - E(X_2)) + \rho(X_1 + X_2) - E(X_1 + X_2) - (\rho(X_1) - E(X_1)) \\
&= 2\rho(X_1 + X_2) - E(X_1 + X_2) - (\rho(X_1) + \rho(X_2)) \\
&\leq 2\rho(X_1 + X_2) - E(X_1 + X_2) - \rho(X_1 + X_2) \text{ (by subadditivity)} \\
&= \rho(X_1 + X_2) - E(X_1 + X_2) \\
&= C
\end{aligned}
$$

If there are three underwriting divisions, apply the logic in the above proof to (X_1+X_2) and to X_3. Next use the result from the proof directly on (X_1+X_2) to get the final result for three underwriting divisions.

Proceed inductively to get the result for 4, 5, ... underwriting divisions.

■

Since it is the job of insurers to diversify risk, the inequality of Proposition 5.2 should be strict. That is, the sum of the marginal capitals should be strictly less than the total capital. That requirement leads us to the following proposition.

Proposition 5.3

If the sum of the marginal capitals is less than the total capital, and the insurer expects to make a return, $e = P/C$, then at least some of its underwriting divisions must have a return on its marginal capital greater than e.

Proof

Assume that $\frac{\Delta P_k}{\Delta C_k} = \frac{P}{C} \equiv e$ for all underwriting divisions, k. Then:

$$P = \sum_k \Delta P_k = \frac{P}{C} \sum_k \Delta C_k < P \quad (!)$$

This contradiction means that we must have $\Delta P_k / \Delta C_k > e$ for at least one k.

■

Suppose an insurer has a choice of continuing its business in one of two underwriting divisions j and k. In its analysis of market prices, the insurer finds that it can expect to make profits of ΔP_j and ΔP_k for underwriting divisions j and k, respectively. Furthermore, it calculates that it must retain ΔC_j and ΔC_k of capital for underwriting divisions j and k, respectively. From a financial point of view, it makes sense for the insurer to favor the underwriting division that has the larger return on marginal capital.

Over time, each underwriting division in the company will come under similar scrutiny, with the ultimate result that each underwriting division will expect the same return, d, on marginal capital.

Let A_k be the capital allocated to the underwriting division k. Then:

$$\frac{\Delta P_k}{A_k} = e \text{ and } \frac{\Delta P_k}{\Delta C_k} = d. \tag{5.1}$$

Hence $\sum_j \Delta P_j = d \sum_j \Delta C_j = e \sum_j A_j = eC$, and thus $e = d \frac{\sum_j \Delta C_j}{C}$. (5.2)

Substituting Equation 5.1 into Equation 5.2 yields:

$$\frac{\Delta P_k}{A_k} = \frac{\Delta P_k}{\Delta C_k} \frac{\sum_j \Delta C_j}{C}. \tag{5.3}$$

Solving Equation 5.3 for A_k yields:

$$A_k = \Delta C_k \frac{C}{\sum_j \Delta C_j}. \tag{5.4}$$

We now recap the chain of reasoning in this section.

1. We started with the assumption that we want to derive an insurer's required capital from a subadditive measure of risk. A subadditive measure of risk is desirable because it reflects the benefits of diversification.

2. The policyholders must ultimately bear the cost of providing the insurer's capital. How much of that cost each policyholder must bear becomes an issue. In this paper, I have chosen to allocate the cost to insurer defined underwriting divisions. (In principle, the underwriting divisions could be individual policyholders.)

3. The method I have chosen to allocate the cost of capital to the underwriting divisions is to allocate the insurer's capital to underwriting divisions and then apply the insurer's selected rate of return to the allocated capital. (I chose the capital allocation method because it is conventional and not because it is fundamentally necessary.)

4. Proposition 5.1 limits our choice of capital allocation methods. If we require an underwriting division to "carry its own weight," the capital allocated to the underwriting division can be no less than its marginal capital.

5. Proposition 5.3 tells us that setting the allocated capital equal to the marginal capital will not lead to the insurer's recovering its cost of capital from the underwriting divisions.

6. We make the additional assumption that in the long run, insurers will structure their books of business so that their return on marginal capital is the same for all underwriting divisions. That assumption leads to a capital allocation formula, Equation 5.4, that amounts to multiplying the marginal capital for each underwriting division by a constant factor.

I should point out that other long run assumptions, such as those made by Game Theory, lead to different capital allocation formulas. See Delbaen and Denault [5], and Mango [7] for additional details.

6. Allocating Capital to Support Outstanding Loss Reserves

The insurer's pledge to pay losses can be a long-term commitment. As time goes on, the insurer pays some losses and the uncertainty in future loss payments declines. Therefore the insurer can release some of the original capital allocated to an underwriting division, for a given accident year, can be released.

In the current year, the insurer will have its capital supporting the outstanding losses from prior accident years. In this section, we apply the logic described in Section 5 and allocate capital to outstanding loss reserves. We calculate the reduction in needed capital when the outstanding losses are removed from the insurance company, and then allocate the capital in proportion to the marginal capital of each underwriting division and each loss reserve. Keep in mind that when establishing target rates of return for the current year, we must consider how much capital the insurer will allocate to the outstanding losses in future years. To do that, the insurer needs a plan for its future business.

Allocation of capital has been actively discussed in the Casualty Actuarial Society over the past several years. The classic "Kenney Rule" was a rule of thumb for capital adequacy. It simply stated that an insurer was adequately capitalized if its premium to capital ratio was two to one. Insurers could easily apply such a rule by line of insurance by setting the allocated capital supporting an underwriting division by dividing its premium by two.

A problem with such an allocation is that it does not recognize variability in the length of time, by line of insurance, that insurers must hold capital. Russell Bingham [4] recognizes that problem. His solution is to allocate capital in proportion to the reserve to capital ratio. That allocation is a step in the right direction. One might expect that a larger reserve would indicate a larger uncertainty in the reserve, and hence the insurer should allocate more capital to the larger reserve. However, the size of the reserve might not be proportional to the risk it contributes to the insurer. Consider the case where the insurer knows for certain that it will have to pay a fixed amount A at some time t in the future. The insurer sets up a loss reserve for this fixed amount A but needs no additional capital to support it. Conversely, suppose the insurer will have to pay a claim of an uncertain amount at time t in the future. Suppose further that the expected payment is equal to A. The insurer sets up a loss reserve equal to this expected amount, A, but will have to hold additional capital because of the uncertain amount of the claim. If the insurer were to allocate capital in proportion to reserves, it would allocate the same amount to each of those claim reserves. The approach I have taken in another paper, Meyers [7], is to use claim severity distributions that vary by settlement lag. That is a further step in the right direction because it recognizes variability in the loss reserve. However, the claim severity distributions, derived from claims settled after a given time, do not recognize the additional information that may be available at the time of the reserve evaluation. Work done by Taylor [9] for the CAS Committee on the Theory of Risk addresses the problem of additional knowledge. That approach may move the problems further toward the ultimate solution.

7. Reinsurance

An insurer can reduce the amount of capital it needs by buying reinsurance. When buying reinsurance, the insurer faces a transaction cost (that is, the reinsurance premium less the provision for expected loss) that replaces a portion of the capital. Note that the insurer does not need to know the reinsurer's pricing assumptions. The insurer can, and perhaps should, use its own estimate of the reinsurer's expected loss to back out the reinsurance transaction cost.

Taxes play an important role in the transaction costs of reinsurance. The insurer deducts reinsurance costs from its taxable income. Capital, whether raised externally or from retained earnings, is subject to corporate income tax. Vaughan [10] points out that the tendency for reinsurance to stabilize insurer income also provides tax advantages. That gives reinsurance an advantage as a provider of insurer financing.

8. The Cost of Financing Insurance

Ultimately, an insurer must be able to pay its financing costs out of the premiums charged to the insureds and from the returns on invested assets. We now determine how much of those financing costs should come from premium. The first step is to decide on a target return on equity. Typically, an insurer's board of directors makes that decision based on considerations described in Section 5.

Investors provide the capital to the insurer. In return, they expect to receive a cash flow reflecting:

1. Premium income
2. Payments to reinsurers
3. Investment income
4. Loss and expense payments
5. Income from the capital that is released as liabilities either expire or become certain

Premium income and payments to reinsurers contain provisions for losses and expenses. It will simplify matters to remove loss and expense payments from our immediate attention by taking expected values and allowing the actual losses in (4) to cancel out the expected loss provisions in (1) and (2). That simplification allows us to concentrate on the cash flow of insurer capital and the net cost of reinsurance, that is, the cost of financing insurance. Investors provide capital to the insurer. After netting out the insurer's loss and expense payments the investors receive a cash flow reflecting:

1. Income from the profit provision in the premium
2. Payments of the net costs to reinsurers
3. Investment income from the capital held for uncertain liabilities
4. Income from the capital that is released as liabilities either expire or become certain

The insurer makes its targeted return on capital if the present value of that cash flow, evaluated *at the targeted return on capital*, is equal to the invested capital. If we allow that:

1. The insurer collects the profit provision in the premium immediately.
2. The insurer makes its reinsurance payments immediately.
3. The insurer determines its necessary capital at the beginning of the year and holds that capital at the end of the year. The insurer then releases capital not needed for the next year. The insurer simultaneously releases investment income on the invested capital.

Then the profit provision necessary for the insurer to make its targeted return on equity is equal to:

Capital + Reinsurance Transaction Costs – Present Value of Released Capital.

To get the profit provision for each underwriting division we need to calculate the marginal cost of capital and the transaction costs for reinsurance for: (1) each underwriting division; and (2) each outstanding loss reserve. We now examine the calculations in some detail.

Table 8.1

Component for Accident Year y	Symbol
Capital investment for current calendar year y+t Note: The insurer needs the capital to cover claims from the current year as well as claims incurred in prior years. The capital also covers business projected for accident years, up to and including year y+t.	$C(t)$
Capital needed in calendar year y+t if the insurer removes underwriting division/accident year k	$C_k(t)$
Marginal Capital for underwriting division/accident year k in calendar year y+t	$\Delta C_k(t) = C(t) - C_k(t)$
Sum of marginal capitals in calendar year y+t	$SM(t)$
Capital allocated to underwriting division/accident year k for calendar year y+t	$A_k(t) = C(t)\Delta C_k(t)/SM(t)$
Transaction costs for underwriting division k's reinsurance (for current accident year only)	$R_k(0)$
Profit provision for underwriting division k	$\Delta P_k(0)$
Insurer's return on its investments	i
Insurer's target return on capital	e

The capital allocated to a given time period earns interest until the beginning of the next period. At that time, the insurer releases a portion of the capital either to pay for losses or to return to the investors.

Table 8.2

Time	Financial Support Allocated at Time t	Amount Released at Time t
0	$A_k(0) + R_k(0)$	0
1	$A_k(1)$	$Rel_k(1) = A_k(0)(1+i) - A_k(1)$
---	---	---
t	$A_k(t)$	$Rel_k(t) = A_k(t-1)(1+i) - A_k(t)$
---	---	---

Then:

$$\Delta P_k(0) = \underbrace{A_k(0) - \sum_{t=1}^{\infty} \frac{Rel_k(t)}{(1+e)^t}}_{\text{Cost of Capital}} + \underbrace{R_k(0)}_{\substack{\text{Net Cost of} \\ \text{Reinsurance}}} . \qquad (8.1)$$

Equation 8.1 gives the profit provision for underwriting division k.

I selected $\alpha = 99\%$ as the threshold to determine the ABC Insurance Company's capital using the Tail Value at Risk. I selected $T = 2.185$ as the multiple using the Standard Deviation measure of risk. The reason for the odd multiple, T, is that it will force equality in the necessary capital for two examples given below. The basic loss statistics are given in Table 3.1. I applied a covariance generator, $b = 0.03$, to the non-catastrophe losses.

Tables 8.3 and 8.4 show the results of the capital allocation calculations for the Tail Value at Risk (TVaR) measure and the Standard Deviation measure of risk respectively. (Note that for the Standard Deviation measure of risk, the allocation percentages are the same no matter what multiplier is used. So I omitted the multiplier in the calculations.)

Note that we allocate capital to outstanding losses from prior years. In future years, we will need to allocate capital to outstanding losses from the current year. And we must fund the cost of that capital from the current year's premiums. The capital allocated to outstanding losses in future years will, in part, depend upon future writings. To keep it simple (and to save paper) I assumed that future writings are the same as past writings.

Table 8.3

Capital Allocation Calculation for Tail Value at Risk

Calendar Year 2002 Line & AY	E[OS Loss] w/o Line & AY	VaR[OS Loss] w/o Line & AY	TVaR[OS Loss] w/o Line & AY	Marginal TVaR Capital	% Allocated Capital
GL-1998	475,000,000	720,000,512	773,855,722	206,015	0.118%
GL-1999	467,000,000	711,998,997	764,994,608	1,067,129	0.610%
GL-2000	452,000,000	697,000,933	748,373,602	2,688,136	1.537%
GL-2001	432,000,000	677,000,063	726,214,789	4,846,948	2.771%
GL-2002	407,000,000	651,999,362	698,687,861	7,373,876	4.216%
PL-1998	472,000,000	716,999,867	770,515,190	546,547	0.312%
PL-1999	462,000,000	706,999,494	759,373,602	1,688,136	0.965%
PL-2000	447,000,000	691,999,076	742,630,697	3,431,041	1.962%
PL-2001	427,000,000	671,999,821	720,525,337	5,536,401	3.165%
PL-2002	407,000,000	652,000,766	698,381,454	7,680,283	4.391%
Auto-2000	467,000,000	712,000,685	765,021,207	1,040,530	0.595%
Auto-2001	442,000,000	687,000,559	737,398,147	3,663,590	2.095%
Auto-2002	407,000,000	652,000,474	698,804,347	7,257,390	4.149%
Prop-2002	442,000,000	687,000,812	737,354,017	3,707,720	2.120%
Cat-2002	472,000,000	639,672,796	646,894,524	124,167,213	70.993%
Combined/Total	**477,000,000**	**721,999,255**	**776,061,737**	**174,900,954**	**100.000%**

At this point, it will be helpful to connect the equations in Table 8.1 with the numbers in Table 8.3. Here are some illustrated calculations.

- Calendar year y = 2002
- Capital needed for calendar year 2002 = $C(0)$ = 776,061,737 – 477,000,000 = 299,061,737.
- Capital needed in calendar year 2002 if we remove (k =) GL underwriting division/accident year 2002 = $C_k(0)$ = 698,687,861 – 407,000,000 = 291,687,861.
- Marginal capital for (k =) GL underwriting division/accident year 2002 is $\Delta C_k(0)$ = 299,061,737 – 291,687,861 = 7,373,876.
- The sum of the marginal capitals, $SM(0)$, is equal to 174,900,954.
- The percentage of capital allocated to (k=) GL underwriting division/accident year 2002 is $\Delta C_k(0)/SM(0)$ = 4.216%.
- At the beginning of calendar year 2002, ABC has unpaid losses from accident year 2001. Following the procedure outlined above, we calculate that the percentage of capital allocated to (k=) GL underwriting division/accident year 2001 = 2.771%.
- Since we are assuming that future writings are the same as past writings, we have that for (k=) GL underwriting division/accident year 2002, $\Delta C_k(1)/SM(1)$ is also equal to 2.771%. If ABC planned to change future writings, we would need an accident year 2003 version of Table 8.3.

Table 8.4 gives the underwriting division/accident year allocations for the Standard Deviation measure of risk.

Table 8.4

Capital Allocation Calculation for Standard Deviation Measure of Risk

Calendar Year 2002 Line & AY	**E[OS Loss] w/o Line & AY**	**Std[OS Loss] w/o Line & AY**	**Marginal Std[OS Loss]**	**% Allocated Capital**
GL-1998	475,000,000	89,571,750	316,618	0.387%
GL-1999	467,000,000	88,297,121	1,591,247	1.947%
GL-2000	452,000,000	85,915,067	3,973,301	4.862%
GL-2001	432,000,000	82,760,946	7,127,422	8.721%
GL-2002	407,000,000	78,907,221	10,981,147	13.436%
PL-1998	472,000,000	89,088,446	799,922	0.979%
PL-1999	462,000,000	87,479,134	2,409,235	2.948%
PL-2000	447,000,000	85,067,393	4,820,976	5.899%
PL-2001	427,000,000	81,933,929	7,954,439	9 733%
PL-2002	407,000,000	78,817,625	11,070,744	13 546%
Auto-2000	467,000,000	88,304,587	1,583,782	1.938%
Auto-2001	442,000,000	84,364,647	5,523,722	6.759%
Auto-2002	407,000,000	78,942,392	10,945,976	13.393%
Prop-2002	442,000,000	84,351,933	5,536,435	6.774%
Cat-2002	472,000,000	82,794,437	7,093,932	8.680%
Combined/Total	**477,000,000**	**89,888,369**	**81,728,899**	**100.000%**

Perhaps the more striking comparison between the measures of risk is in the capital allocated to the catastrophe underwriting division.

We now continue the calculations described in Table 8.1 and 8.2.

Table 8.5

Needed Tail Value at Risk Allocated Capital at the Beginning of Each Calendar Year for Accident Year 2002

Line\Cal Year	2002	2003	2004	2005	2006
General Liability	12,608,532	8,287,757	4,596,421	1,824,675	352,263
Products Liability	13,132,455	9,466,647	5,866,709	2,886,530	934,536
Auto	12,409,354	6,264,344	1,779,193	0	0
Property	6,339,801	0	0	0	0
Catastrophe	212,312,521	0	0	0	0
Other OS Losses	42,259,075	275,042,989	286,819,413	294,350,533	297,774,938
TVaR Capital	299,061,737	299,061,737	299,061,737	299,061,737	299,061,737

We continue the illustrative calculations.

- The capital allocated to (k=) GL underwriting division/accident year 2002, $A_k(0)$, is equal to the total capital for calendar year 2002, (299,061,737), times the corresponding allocation percentage from Table 8.3, (4.216%) and is equal to 12,608,532.
- The capital allocated to (k=) GL underwriting division/accident year 2002, $A_k(1)$, is equal to the total capital for calendar year 2003, (299,061,737), times the corresponding allocation percentage from Table 8.3, (2.771%) and is equal to 8,287,757.
- Other OS Losses refers to outstanding losses from other accident years.

Table 8.6 gives the capital allocations for the Standard Deviation Measure of risk.

Table 8.6

Needed Standard Deviation Allocated Capital at the Beginning of Each Calendar Year for Accident Year 2002

Line\Cal Year	2002	2003	2004	2005	2006
General Liability	26,387,924	17,127,344	9,547,925	3,823,801	760,840
Products Liability	26,603,226	19,114,682	11,584,905	5,789,442	1,922,229
Auto	26,303,408	13,273,618	3,805,861	0	0
Property	13,304,169	0	0	0	0
Catastrophe	17,046,865	0	0	0	0
Other OS Losses	86,750,647	146,880,595	171,457,548	186,782,997	193,713,170
Std Dev Capital	196,396,239	196,396,239	196,396,239	196,396,239	196,396,239

- The total capital for the Standard Deviation measure of risk, 196,396,239, is given by the standard deviation, 89,888,369, (from Table 8.4) times our selected multiplier, 2.185.

The next step is to calculate how much capital the insurer can release at the end of each year.

For each underwriting division, the insurer:

1. Receives allocated capital (Tables 8.5 and 8.6)
2. Earns interest on that capital (here assumed to be 6%)
3. Releases capital not needed for the next year

Tables 8.7 and 8.8 give the results of those calculations.

Table 8.7

Schedule for Releasing Tail Value at Risk Capital at the End of Each Calendar Year for Accident Year 2002

Line\Cal Year	2002	2003	2004	2005	2006
General Liability	5,077,287	4,188,601	3,047,532	1,581,892	373,399
Products Liability	4,453,755	4,167,937	3,332,182	2,125,185	990,609
Auto	6,889,571	4,861,011	1,885,945	0	0
Property	6,720,189	0	0	0	0
Catastrophe	225,051,272	0	0	0	0

Here is a sample calculation:

- The amount of capital released for General Liability at the end of 2002 is equal to 12,608,532 *times* 1.06 *minus* 8,287,757 = 5,077,287.

Table 8.8

Schedule for Releasing Standard Deviation Capital at the End of Each Calendar Year for Accident Year 2002

Line\Cal Year	2002	2003	2004	2005	2006
General Liability	10,843,856	8,607,059	6,297,000	3,292,388	806,491
Products Liability	9,084,738	8,676,658	6,490,557	4,214,579	2,037,563
Auto	14,607,994	10,264,174	4,034,213	0	0
Property	14,102,419	0	0	0	0
Catastrophe	18,069,676	0	0	0	0

Now that we have calculated the schedule for releasing capital, we can then apply Equation 8.1 to calculate the cost of capital (that is, profit) that must be supplied by the policyholders. We set e = 12.00%. Here are the results:

Table 8.9

Cost of Capital by Underwriting Division

	TVaR Capital	Std Dev Capital
General Liability	1,349,742	2,812,338
Products Liability	1,548,761	3,120,415
Auto	1,040,404	2,206,546
Property	339,632	712,723
Catastrophe	11,373,885	913,225
Total	15,652,425	9,765,247

Note the relative size of the catastrophe cost of capital in the two measure of risks.

9. The Cost of Financing Insurance When Using Reinsurance

We have seen that, the effect of reinsurance is to replace part of the cost of capital with the net cost of reinsurance. In this section, we will apply the equations in Tables 8.1 and 8.2 to the ABC Insurance Company when it purchases catastrophe insurance covering losses in excess of $50 million.

Insurers deduct the cost of reinsurance, including the reinsurer's expenses and profit, from taxable income. The net cost of the reinsurance is then equal to:

$$\text{Expected Reinsurance Recovery} \times \left(\frac{1}{ELR} - 1 \right) \times (1 - \text{Tax Rate}),$$

where ELR is the reinsurer's expected loss ratio. I set the tax rate equal to 35%.

As in the last section, we now calculate the total cost of financing ABC's insurance portfolio for the two measures of risk. The following tables, corresponding to the tables in Section 8, show the calculations with catastrophe reinsurance.

Table 9.1

Capital Allocation Calculation for Tail Value at Risk with Catastrophe Reinsurance

Calendar Year 2002 Line & AY	E[OS Loss] w/o Line & AY	VaR[OS Loss] w/o Line & AY	TVaR[OS Loss] w/o Line & AY	Marginal TVaR Capital	% Allocated Capital
GL-1998	471,000,000	639,782,921	651,919,381	622,782	0.386%
GL-1999	463,000,000	629,143,638	641,288,266	3,253,897	2.015%
GL-2000	448,000,000	609,167,042	621,332,646	8,209,517	5.083%
GL-2001	428,000,000	582,538,754	594,738,644	14,803,519	9.166%
GL-2002	403,000,000	549,931,480	562,138,364	22,403,799	13.872%
PL-1998	468,000,000	635,724,080	647,866,024	1,676,139	1.038%
PL-1999	458,000,000	622,167,113	634,332,646	5,209,517	3.226%
PL-2000	443,000,000	601,704,202	613,925,904	10,616,259	6.573%
PL-2001	423,000,000	575,284,274	587,535,781	17,006,382	10.530%
PL-2002	403,000,000	548,708,303	561,006,009	23,536,154	14.573%
Auto-2000	463,000,000	629,247,167	641,388,772	3,153,392	1.952%
Auto-2001	438,000,000	596,263,226	608,424,852	11,117,311	6.883%
Auto-2002	403,000,000	550,393,465	562,573,065	21,969,098	13.602%
Prop-2002	438,000,000	596,089,957	608,259,153	11,283,010	6.986%
Cat-2002	472,000,000	639,672,796	646,894,524	6,647,640	4.116%
Combined/Total	**473,000,000**	**642,406,295**	**654,542,163**	**161,508,417**	**100.000%**

Table 9.2

Capital Allocation Calculation for Standard Deviation with Catastrophe Reinsurance

Calendar Year 2002 Line & AY	E[OS Loss] w/o Line & AY	Std[OS Loss] w/o Line & AY	Marginal Std[OS Loss]	% Allocated Capital
GL-1998	471,000,000	82,747,196	342,628	0.419%
GL-1999	463,000,000	81,365,727	1,724,096	2.110%
GL-2000	448,000,000	78,774,354	4,315,470	5.281%
GL-2001	428,000,000	75,321,804	7,768,019	9.505%
GL-2002	403,000,000	71,065,812	12,024,012	14.713%
PL-1998	468,000,000	82,223,788	866,036	1.060%
PL-1999	458,000,000	80,477,319	2,612,505	3.197%
PL-2000	443,000,000	77,848,965	5,240,859	6.413%
PL-2001	423,000,000	74,412,155	8,677,669	10.618%
PL-2002	403,000,000	70,966,316	12,123,508	14.835%
Auto-2000	463,000,000	81,373,829	1,715,995	2.100%
Auto-2001	438,000,000	77,080,436	6,009,388	7.353%
Auto-2002	403,000,000	71,104,861	11,984,962	14.665%
Prop-2002	438,000,000	77,066,521	6,023,303	7.370%
Cat-2002	472,000,000	82,794,437	295,387	0.361%
Combined/Total	**473,000,000**	**83,089,824**	**81,723,836**	**100.000%**

Table 9.3
Needed Tail Value at Risk Allocated Capital with Catastrophe Reinsurance at the Beginning of Each Calendar Year for Accident Year 2002

Line\Cal Year	2002	2003	2004	2005	2006
General Liability	25,182,800	16,639,770	9,227,838	3,657,516	700,033
Products Liability	26,455,614	19,115,879	11,933,116	5,855,714	1,884,050
Auto	24,694,178	12,496,319	3,544,543	0	0
Property	12,682,572	0	0	0	0
Catastrophe	7,472,223	0	0	0	0
Other OS Losses	85,054,777	133,290,195	156,836,667	172,028,934	178,958,080
TVaR Capital	181,542,163	181,542,163	181,542,163	181,542,163	181,542,163

Table 9.4
Needed Standard Deviation Allocated Capital with Catastrophe Reinsurance at the Beginning of Each Calendar Year for Accident Year 2002

Line\Cal Year	2002	2003	2004	2005	2006
General Liability	26,710,263	17,255,957	9,586,429	3,829,925	761,117
Products Liability	26,931,284	19,276,662	11,642,098	5,803,446	1,923,821
Auto	26,623,518	13,349,315	3,811,929	0	0
Property	13,380,227	0	0	0	0
Catastrophe	656,175	0	0	0	0
Other OS Losses	87,240,698	131,660,229	156,501,708	171,908,793	178,857,226
Std Dev Capital	181,542,163[1]	181,542,163	181,542,163	181,542,163	181,542,163

Table 9.5
Schedule for Releasing Tail Value at Risk Capital with Catastrophe Reinsurance at the End of Each Calendar Year for Accident Year 2002

Line\Cal Year	2002	2003	2004	2005	2006
General Liability	10,053,998	8,410,318	6,123,993	3,176,933	742,035
Products Liability	8,927,071	8,329,716	6,793,389	4,323,006	1,997,093
Auto	13,679,509	9,701,556	3,757,216	0	0
Property	13,443,526	0	0	0	0
Catastrophe	7,920,556	0	0	0	0

Table 9.6
Schedule for Releasing Standard Deviation Capital with Catastrophe Reinsurance at the End of Each Calendar Year for Accident Year 2002

Line\Cal Year	2002	2003	2004	2005	2006
General Liability	11,056,921	8,704,886	6,331,690	3,298,604	806,784
Products Liability	9,270,499	8,791,164	6,537,178	4,227,832	2,039,250
Auto	14,871,613	10,338,345	4,040,644	0	0
Property	14,183,040	0	0	0	0
Catastrophe	695,545	0	0	0	0

[1] The Standard Deviation Capital Multiplier of 2.185 was selected so that the capital required for the TVaR capital is equal to the standard deviation capital for the catastrophe reinsurance case.

Table 9.7

The Cost of Financing Insurance with Catastrophe Reinsurance

	Tail Value at Risk Capital			Standard Deviation Capital		
	Cost of Capital	Net Cost of Reinsurance	Cost of Financing	Cost of Capital	Net Cost of Reinsurance	Cost of Financing
General Liability	2,702,376	0	2,702,376	2,837,645	0	2,837,645
Products Liability	3,128,662	0	3,128,662	3,148,768	0	3,148,768
Auto	2,071,998	0	2,071,998	2,227,575	0	2,227,575
Property	679,423	0	679,423	716,798	0	716,798
Catastrophe	400,298	2,600,000	3,000,298	35,152	2,600,000	2,635,152
Total	8,982,757	2,600,000	11,582,757	8,965,938	2,600,000	11,565,938

Compare Table 9.7 with Table 8.9. Note that the cost of financing insurance for ABC decreases with the reinsurance when we measure risk by the Tail Value at Risk, while it increases with this reinsurance when we measure risk by the standard deviation.

Now, anyone familiar with real-world catastrophe reinsurance knows that the price of catastrophe reinsurance can vary widely from time to time. When prices go down, insurers buy more reinsurance, and when prices go up they buy less. That behavior is consistent with this model of ABC Insurance Company. Consider the following tables, where we calculate the level of reinsurance that minimizes the cost of financing insurance.

Table 9.8

Optimal Level of Reinsurance when Risk is Measured by the Tail Value at Risk

Reinsurance ELR	**Optimal Cat Limit**	**Cost of Financing**
50.00%	40,976,282	11,572,039
60.00%	29,012,942	10,639,167
70.00%	21,219,679	9,943,190
80.00%	14,660,548	9,404,707
90.00%	8,136,819	8,974,210

Table 9.9

Optimal Level of Reinsurance when Risk is Measured by the Standard Deviation

Reinsurance ELR	**Optimal Cat Limit**	**Cost of Financing**
50.00%	No Limit	9,765,247
60.00%	No Limit	9,765,247
70.00%	212,024,801	9,748,414
80.00%	119,610,539	9,551,183
90.00%	52,467,114	9,254,743

Although the two measures of risk both indicate, qualitatively, observed reinsurance purchasing behavior, the quantitative results are quite different. The Tail Value at Risk indicates a greater use of catastrophe reinsurance – for ABC Insurance Company. My own sense of the reinsurance market leads me to hypothesize that the Tail Value at Risk will provide a better explanation of reinsurance market behavior. Research could test my hypothesis by applying this methodology to real insurers and seeing to what extent insurers follow the indicated behavior.

10. Target Combined Ratios

All that remains is to express our results in terms of target combined ratios for the ABC Insurance Company. To do that, we need to make the following additional expense assumptions.

Table 10.1

Underwriting Expense Factors

Underwriting Division	ULAE % of Loss	Other Expense % of Premium
General Liability	10.00%	30.00%
Product Liability	10.00%	30.00%
Auto	7.00%	30.00%
Property	7.00%	30.00%
Catastrophe	7.00%	30.00%

We also need the actuarial present value (APV) of the losses for each of the underwriting divisions.

Table 10.2

Underwriting Division	Expected Loss	APV of Loss
General Liabililty	70,000,000	63,637,691
Products Liability	70,000,000	62,720,330
Auto	70,000,000	65,547,100
Property	35,000,000	33,995,005
Catastrophe	5,000,000	4,856,429
Total	250,000,000	230,756,556

I derived the following target combined ratios using the expense factors from Table 10.1 and the cost of financing insurance from the Tail Value at Risk part of Table 9.7.

Table 10.3

Target Combined Ratios for Tail Value at Risk with Catastrophe Reinsurance

	General Liability	Products Liability	Auto	Property	Catastrophe
E[Loss]	70,000,000	70,000,000	70,000,000	35,000,000	5,000,000
APV[Loss]	63,637,691	62,720,330	65,547,100	33,995,005	4,856,429
ULAE%	10.00%	10.00%	7.00%	7.00%	7.00%
ULAE	7,000,000	7,000,000	4,900,000	2,450,000	350,000
APV of LAE	6,363,769	6,272,033	4,588,297	2,379,650	339,950
Other Expense%	30.00%	30.00%	30.00%	30.00%	30.00%
Other Expense	31,158,787	30,909,011	30,946,026	15,880,320	3,512,862
Cost of Financing	2,702,376	3,128,662	2,071,998	679,423	3,000,298
Cost of Financing%	2.60%	3.04%	2.01%	1.28%	25.62%
Premium	103,862,622	103,030,037	103,153,422	52,934,399	11,709,539
Target Comb Ratio	104.14%	104.74%	102.61%	100.75%	75.69%
Overall Comb Ratio	102.51%				

Lest we forget – in Section 3, I stressed the importance of correlation. Recall that we generated correlations in the noncatastrophe underwriting divisions using random β's with variance $b = 0.03$. Changing $b = 0.03$ to $b = 0.01$ reduces the overall needed capital from 181,542,163 to 119,199,301. The following table gives the corresponding changes in the target combined ratios.

Table 10.4

Target Combined Ratios for
Tail Value at Risk with Catastrophe Reinsurance and $b = 0.01$

	General Liability	Products Liability	Auto	Property	Catastrophe
E[Loss]	70,000,000	70,000,000	70,000,000	35,000,000	5,000,000
APV[Loss]	63,637,691	62,720,330	65,547,100	33,995,005	4,856,429
ULAE%	10.00%	10.00%	7.00%	7.00%	7.00%
ULAE	7,000,000	7,000,000	4,900,000	2,450,000	350,000
APV of LAE	6,363,769	6,272,033	4,588,297	2,379,650	339,950
Other Expense%	30.00%	30.00%	30.00%	30.00%	30.00%
Other Expense	30,731,258	30,433,530	30,610,823	15,772,387	3,542,252
Cost of Financing	1,704,808	2,019,207	1,289,858	427,582	3,068,875
Cost of Financing%	1.66%	1.99%	1.26%	0.81%	25.99%
Premium	102,437,525	101,445,101	102,036,078	52,574,625	11,807,507
Target Comb Ratio	105.17%	105.90%	103.41%	101.23%	75.31%
Overall Comb Ratio	103.37%				

11. Concluding Remarks

In constructing the example for ABC Insurance Company, I made two important simplifications that were not mentioned above. First, I did not consider asset risk. And second, I minimized the effort in the selection of solvency thresholds.

In our exercise, ABC assets earned a fixed rate of interest of 6%. If ABC invested in higher-yielding assets with variable returns, the company would have to have more assets, and hence more capital. That observation suggests a need to allocate capital between the underwriting and investment operations. I suggest making such an allocation by first calculating how much capital the company requires with fixed-rate investments, and then calculating how much capital the company requires with the actual investments. The difference between the two will yield the marginal capital for the investment operation.

The most influential determinants of insurer capital are the state regulators and the rating agencies. To take a first crack at determining a solvency threshold, we could determine appropriate capital by consulting regulators and rating agencies. We would then back the threshold out of that capital.

If we were to do the exercise on several insurers, we should then be able to reach a consensus on the appropriate threshold.

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