Insurance Risk-Based Capital with a Multi-Period Time Horizon

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Abstract: There are two competing views on how to determine capital for an insurer whose loss liabilities extend for several time periods until settlement. The first focuses on the immediate period (usually one-year) and the second uses the runoff (until ultimate loss payment) time frame; each method will generally produce different amounts of required capital. Using economic principles, this study reconciles the two views and provides a general framework for determining capital for multiple periods.

For an insurer whose liabilities and corresponding assets extend over a single time period, Butsic [2013] determined the optimal capital level by maximizing the value of the insurance to the policyholder, while providing a fair return to the insurer's owners. This paper extends those results to determine optimal capital when liabilities last for several time periods until settlement. Given the optimal capital for one period, the analysis applies backward induction to find optimal capital for successively longer time frames.

A key element in this approach is the stochastic process for loss development; another is the choice of capital funding strategy, which must respond to the evolving loss estimate. In addition to the variables that affect the optimal one-period capital amount (such as the loss volatility, frictional cost of capital and the policyholder risk preferences), in this paper I show that the horizon length, the capitalization interval (time span between potential capital flows), and the policy term will influence the optimal capital for multiple time periods. Institutional and market factors, such as the conservatorship process for insolvent insurers and the cost of raising external capital, also play a major role and are incorporated into the model.

Results show that the optimal capital depends on *both* the annual and the ultimate loss volatility. Consequently, more total capital (ownership plus policyholder-supplied capital) is required as the time horizon increases; however, optimal ownership capital may *decrease* as the time horizon lengthens due to the policyholder-supplied capital, which includes premium components for risk margins and income taxes. Also, less capital is needed if capital flows can occur frequently and/or if the policy term is shorter. Insurers that are able to more readily raise capital externally will need to carry less of it.

The model is extended to develop asset risk capital and incorporate features, such as present value and risk margins, that are necessary for practical applications. Although the primary focus is property-casualty insurance, the method can be extended to life and health insurance. In particular, the approach used to determine capital required for multi-period asset risk will apply to these firms.

The resulting optimal capital for insurers can form the basis for pricing, corporate governance and regulatory applications.

Keywords: Backward induction, capital strategy, capitalization interval, certainty-equivalent loss, conservatorship, exponential utility, fair-value accounting, policy term, risk margin, stochastic loss process, technical insolvency, time horizon

1. INTRODUCTION AND SUMMARY

There is a considerable body of literature on how to determine the appropriate risk-based capital for an insurance firm. Generally, the analysis applies a particular risk measure (such as VaR or expected policyholder deficit), calibrated to a specific valuation level (e.g., VaR at 99.5%) to determine the proper amount of capital. However, most of the commonly-used risk measures apply most readily to short-duration risks, for example, property insurance, where the liabilities are settled within a single time period. Application of these methods is more problematic when addressing long-term insurance claims, such as liability, workers compensation and life insurance.

How to treat long-term, or multi-period, liabilities and assets is the subject of much debate in the actuarial and insurance finance literature. For a good, practically-oriented discussion of this topic, see Lowe et al [2011]. Essentially there are two camps: one side advocates using an annual[1] (one-period) time horizon, wherein the current capital amount must be sufficient to offset default risk based on loss liability and asset values over the *upcoming* period, usually one year. The other side argues that the current capital must offset the default over the *entire duration* (the runoff horizon) required to settle the liability. Essentially, the issue is whether capital depends on the loss volatility only for the upcoming year, or the ultimate loss volatility. This controversy has gained momentum with the impending implementation of the Solvency II risk-based capital methodology, which uses an annual (single-period) time horizon.[2]

As shown in the subsequent analysis, the problem may be solved by extending the one-period model to a longer time frame. I have used the concept of an *optimal capital strategy* to determine the appropriate capital amount for the current period, which is the first period of a multi-period liability. For a one-period liability, there is a theoretically optimal amount of capital that depends on the insurer's cost of holding capital and the nature of the policyholders' risk aversion. These results are derived in *An Economic Basis for Property-Casualty Insurance Risk-Based Capital Measurement* (Butsic [2013]), which develops the appropriate risk measure (adjusted ruin, or default probability) and

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[1] More generally, the period could be shorter than one year, but most applications use the annual time frame. In this paper I use the more general concept of time *periods*.
[2] See the European Parliament Directive [2009]; Article 64.

calibration method (using the frictional cost of capital) for *a one-period* insurer in an equilibrium insurance setting. The analysis here can be considered as an extension to this paper which, for reference, I shorten to EBRM.

With multi-period risks, we can use the same fundamental assumptions that drive optimal capital for a single period. The main point is that, as in a one-period model, the optimal capital over several periods depends on the balance between capital costs and the amount that the policyholders are willing to pay to reduce their perceived value of default.

Capital in this paper is defined in the general accounting sense as the difference between assets and liabilities. For practical applications, capital will need to be defined according to a standard accounting convention such as IFRS,[3] U.S. statutory accounting or the accounting used in Solvency II.

Although the analysis is geared toward producing optimal capital for property-casualty insurance *losses*, the methodology also applies to long-term asset risk and life insurance (see sections 8 and 9).

1.1 Summary

The main result of this paper is that the optimal capital for an insurer with multi-period losses depends on *both* the volatility of losses for the current year and the volatility of the ultimate loss value. The ultimate loss volatility is a factor because, when an insurer becomes insolvent, it generally enters conservatorship and the losses will develop further, as if the insurer had remained solvent. This further development depends on the *ultimate* loss volatility. As long as there is volatility for remaining loss development, the optimal total capital (defined as ownership plus policyholder-supplied capital) increases as the time horizon lengthens, but at a decreasing rate. However, because

[3] In IFRS (International Financial Reporting Standards) and Solvency II accounting, the value of unpaid claim liabilities is treated as the best estimate of the unpaid claims plus a risk margin. Sections 2-7 treat liabilities as the best estimate of unpaid claims. The effect of risk margins is discussed in Section 8.

policyholder-supplied capital (needed to pay future capital costs and the risk margin) is included in premiums, and these also increase with loss volatility, the optimal amount of ownership capital may *decrease* if the time horizon is long enough. The ownership capital (e.g., statutory surplus or shareholder equity on an accounting basis) is normally the relevant quantity used for risk-based capital analysis.

For a multiple-period time horizon, the amount of optimal capital depends on the same variables as for an insurer with a single-period horizon: the frictional cost of holding capital (primarily the cost of double-taxation), the degree of policyholder risk aversion, loss/asset volatility and guaranty fund participation. However, with multiple periods, optimal capital also depends on

1. The underlying stochastic process for loss development; the horizon length is also a random variable.
2. What happens to unpaid losses when an insolvency occurs? In particular, conservatorship for an insolvent insurer has a strong effect.
3. The capital strategy used by the insurer. The ability to add capital when needed is particularly important.
4. The cost of raising external capital. In the case of some mutual insurers or privately-held insurers, the limitation on the *ability* to raise capital is a key factor.
5. The length of time between capital flows. The shorter this time frame, the less capital is needed.
6. The policy term. More capital is needed for a longer term, since if default occurs early in the term, the remaining coverage must be repurchased.

Also, the optimal capital depends on two factors important for multi-period risk that are not modeled (for simplicity) in EBRM:

1. The interest rate. As the interest rate increases, less capital is necessary to mitigate default that will occur in the future.
2. The risk margin (or market price of risk) embedded in the premium. This amount acts as policyholder-supplied capital and reduces the amount of ownership capital needed.

As identified in items 3 through 5, optimal capital depends on the insurer's ability to raise capital and the cost of doing so. A lower cost of raising capital and/or better ability to raise capital will imply a lower amount of optimal capital. For most insurers, the best feasible strategy is to add capital when it will improve policyholder welfare, and withdraw capital otherwise. This strategy of adding capital where appropriate (called AC) means that capital is added only if the insurer remains solvent. An alternative strategy (full recapitalization, or FR), adds capital even when the insurer is insolvent. Under FR, only the current-period loss volatility is considered and thus is consistent with the Solvency II risk-based capital methodology.[4] However, the FR strategy is not feasible, so the Solvency II method can understate risk-based capital for long-horizon losses.

The optimal capital for an insurer with *asset risk* is determined by combining the asset risk with the loss risk, and getting the joint capital for both. The implied amount of asset-risk capital is obtained by subtracting the loss-only optimal capital from the joint capital. If the asset risk is low, it is possible that the optimal capital for the combined risks is lower than that for the loss-only risk. Two factors tend to reduce the optimal implied asset-risk capital for long time horizons, compared to the loss-only risk capital. First, when an insurer becomes technically insolvent, asset risk is virtually eliminated, as a consequence of entering conservatorship (where the insurer's investments are replaced with low-risk securities). Second, the positive expected return from risky assets acts as additional capital. As with losses, the optimal asset-risk total capital increases with the time horizon length.

1.2 Outline

The remainder of the paper is summarized thusly:

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[4] The Solvency II approach to risk margins and capital adequacy can be interpreted as assuming that recapitalization is always possible. Note however, that the Solvency II approach includes liability risk margins that increase the amount of assets required of the insurer. These assets increase with the horizon length. The additional (policyholder-supplied) capital from those assets depends on the *ultimate* loss volatility, so that the Solvency II method does not rely solely on the current-period loss volatility. Other than the risk margin issue, I do not compare the Solvency II assumptions to those of the models developed in this paper.

Key Results from the One-Period Model (Section 2)

Section 2 summarizes the results for a one-period model, showing how the cost of holding capital and the policyholder risk preferences will provide an optimal capital amount. Coupled with the insurer's capital strategy, the one-period optimal capital amounts will generate optimal capital for longer-duration losses spanning multiple periods.

Multi-Period Model Issues (Section 3)

Section 3 introduces issues presented in a multi-period model that are not applicable to the one-period case. These issues are explored further in subsequent sections. A key concept is the stochastic loss development process, wherein the estimate of the ultimate loss fluctuates randomly from period to period, with the current estimate being the mean of the ultimate loss distribution; this process determines expected default values in future periods. Another important issue is the impact on assets and loss liabilities following technical insolvency, where a regulator forces an insurer to cease operations when its assets are less than its liabilities; in this case, losses continue to develop after the insurer has defaulted. I describe capital funding strategies, which are necessary to address the period-to-period loss evolution. This section also discusses the distinction between ownership capital and policyholder-supplied capital; this issue may not be relevant in a one-period model.

Basic Multi-period Model (Section 4)

Section 4 presents a basic model of an insurer with multiple-period losses for liability insurance. First, I summarize the assumptions underlying a one-period model and add those necessary for a multi-period model. Then I describe characteristics of the loss development stochastic process, including a parallel certainty-equivalent process needed to value the default from the policyholders' perspective. Third, I specify a premium model, which allows the calculation of the value of the insurance contract to both policyholders and the insurer, and thus the optimum capital amount for both parties. Fourth, I examine the distinction between ownership capital and total capital, which also includes policyholder-supplied capital.[5] Fifth, I discuss capital funding strategies, where insurers attempt to add or withdraw capital to maintain an optimal position over time; the strategies vary according to efficiency (value to policyholders) and feasibility. Finally, I show that the most efficient feasible strategy is where capital is added if the insurer remains solvent; this is denoted as AC.

Optimal Two-period Capital (Section 5)

Section 5 determines the optimal capital for a two-period model under the AC strategy. Here I evaluate the certainty-equivalent value of default under technical insolvency, which is a key component of the analysis. This section introduces a stochastic loss process with normally-distributed incremental development, used in subsequent sections to illustrate optimal capital calculation. Next, the AC model is enhanced to incorporate an additional cost of providing capital from external sources. Finally, I analyze the how optimal capital can be determined for an insurer with limited ability to raise external capital, such as a mutual insurer.

Optimal Capital for More Than Two Periods (Section 6)

Section 6 extends the two-period model to multiple periods using backward induction. This procedure provides optimal initial capital for the various capital strategies.

Capitalization Interval (Section 7)

Section 7 examines how optimal insurer capital depends on the capitalization interval, or the time span required to add capital from external sources. This interval determines the period length for a multi-period model. Section 7 also shows how the policy term affects optimal capital.

Extensions to the Multi-period Model (Section 8)

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[5] For shareholder-owned insurers, policyholder-supplied capital includes the premium components of risk margins and provision for income taxes. In addition to these funds, policyholders of mutual insurers provide ownership capital in their premiums.

Section 8 extends the basic multi-period model to include features necessary for a practical application. I apply a stochastic horizon, where the loss development continues for a random length of time. Also, the analysis shows the effect of using present value and risk margins. The section concludes with a brief discussion of applying the methodology to life and health insurance.

<u>Multi-Period Asset Risk (Section 9)</u>

Section 9 determines optimal capital for asset risk by extending the loss model to a joint loss and asset model. The joint model is simplified by using an augmented loss variable, which incorporates the asset risk and return into a loss-only model.

<u>Conclusion (Section 10)</u>

Section 10 concludes the paper.

<u>Other Material</u>

Appendix A through Appendix D contain detailed numerical examples that illustrate key concepts and provide additional mathematical development. The References provide sources for footnoted information. To assist in following the analysis, the Glossary explains the mathematical notation and abbreviations used in the paper. The final section is a Biography of the Author.

2. KEY RESULTS FROM THE ONE-PERIOD MODEL

This discussion briefly shows how optimum capital is determined in a one-period model. More details can be found in EBRM.

2.1 Certainty-Equivalent Losses

Since a policyholder is presumed to be risk-averse, the perceived value of each possible loss, or claim, amount is different from the nominal value. For a policyholder facing a random loss, the *certainty-equivalent* (CE) value of the loss is the certain amount the policyholder is willing to pay in exchange for removing the risk of the loss. Let L denote the expected value of the loss and $p(x)$ the probability of loss size x. The expected value of the loss is $L = \int_0^\infty xp(x)dx$. The translation from nominal loss amounts to the CE value of the amounts can done using an adjusted probability distribution $\hat{p}(x)$:

$$\hat{L} = \int_0^\infty x\hat{p}(x)\,dx\,. \tag{2.11}$$

Here, $\hat{L}$ is the CE expected loss, with $\hat{L} > L$. The value of the default to the policyholder is called the *certainty-equivalent* expected default (CED) value and is denoted by $\hat{D}$. Its expression is parallel to that of the nominal expected default D:

$$\hat{D} = \int_A^\infty (x - A)\hat{p}(x)\,dx\,. \tag{2.12}$$

Here A is the insurer's asset amount. We have $\hat{D} > D$; for asset values significantly greater than the mean loss L, the CED can be an extremely high multiple of the nominal expected default amount.

If policyholder risk preferences are determined from an expected utility model, then the CE loss distribution can be obtained directly from the unadjusted distribution and the utility function.

2.2 Consumer Value, Capital Costs and Premium

In purchasing insurance, the policyholder pays a premium π in exchange for covering the loss. However, the coverage is only partial, since if the insurer becomes insolvent, only a portion of a loss (claim) is paid. Thus, the value V of the insurance to the policyholder, or *consumer value*, equals the CE loss minus the premium minus the CED, or

$$V = \hat{L} - \pi - \hat{D}\,. \tag{2.21}$$

If $V > 0$, then the policyholder will buy the insurance.

In the basic model described in EBRM (see the assumptions in Section 4) the only costs to the insurer are the loss and the frictional cost of capital (FCC), denoted by z. The FCC is primarily income taxes, but may include principal-agent, regulatory restriction or other costs. Assuming that the capital cost is strictly proportional to the capital amount C, the premium is

$$\pi = L + zC\,. \tag{2.22}$$

Since adding capital reduces the CED but increases premium (through a higher capital cost), there generally will be an optimal level of capital that maximizes V and therefore provides the greatest policyholder welfare. By taking the derivative of V with respect to the asset amount A, we get the requirement for optimal assets, and therefore optimal capital:

$$\hat{Q}(A) = z\,. \tag{2.23}$$

Here $\hat{Q}(A)$ is the default, or ruin, probability under the adjusted probability $\hat{p}(x)$; it equals the negative derivative of $\hat{D}$ with respect to A. This result assumes that the premium is not reduced by the amount of expected default; if so, then equation 2.23 is an approximation.

Meanwhile, the insurer's owners are fairly compensated for the capital cost through the zC component of the premium, so their welfare is also optimized. Since policyholder and shareholder welfare are both maximized, this theoretical optimal capital level can form the basis for pricing, regulation and internal insurer governance.

Notice that if there were no prospect of the insurer's default and the cost of capital were zero, the consumer value of insurance would be the CE expected loss minus the nominal expected loss, or $\hat{L} - L$. Call this amount the *risk value*. It is the maximum possible value that the policyholder could obtain by purchasing insurance. In the basic model, the prospect of default introduces the frictional capital cost and the CE expected default as elements that are subtracted from the risk value to produce the net consumer value. A useful term for the sum of these two amounts is the *solvency cost*. Since the risk value is not a function of the insurer's assets (the basic model assumes riskless assets; risky assets are analyzed in section 9), minimizing the solvency cost is equivalent to maximizing the consumer value.

3. MULTI-PERIOD MODEL ISSUES

Determining optimal capital for multiple periods presents several challenges not evident in the one-period situation. These issues are introduced below and are addressed in greater depth in sections 4 through 9.

3.1 Stochastic Loss Development

In the one-period case, the loss is initially unknown, but its value is revealed at the end of the period. For multiple periods, the loss value may remain unknown for several periods. Consequently, in order to establish the necessary capital amount for each period (using the accounting identity that capital equals assets minus liabilities), we need to estimate the ultimate loss; this assessment is known as the *loss reserve*. The reserve estimate will vary randomly from period to period until the loss is finally settled. The stochastic reserve estimates will form the basis for a dynamic capital strategy.

3.2 Default Definition and Liquidation Management

In a multi-period model, the loss reserve values are *estimates* of the ultimate unpaid loss liability. If the estimated loss exceeds the value of assets at the end of a period, the insurer is deemed to be *technically insolvent*. The insolvency is "technical" because it is possible that the reserve may subsequently develop downward and there is ultimately no default. If the insurer adds sufficient capital to regain solvency, then there is the further possibility that the insurer may yet again become insolvent in future periods. Thus, multiple insolvencies are theoretically possible for a recapitalized individual insurer that emerges from an initial technical insolvency.

Generally, when an insurer becomes technically insolvent, regulators transfer its assets and liabilities to a *conservator*, or receiver, who manages them in the interests of the policyholders. This usually means that the assets are invested conservatively in low-risk securities[6] and when claims are paid, each policyholder gets the same pro-rata share of the assets according to their claim amounts.

There are several important consequences to receivership. First, the liabilities remain "alive" and are allowed to develop further. Second, there is no source of additional capital to mitigate the ultimate default amount (however, no capital can be withdrawn either, unless the assets become significantly larger than the liabilities). Third, the conservative asset portfolio will most likely have a significantly reduced asset risk compared to that of the insurer prior to conservatorship. These features profoundly affect the multi-period capital analysis, as shown in the subsequent sections.

3.3 Dynamic Capital Strategy

In a one-period model the capital is determined once, at the beginning of the period. In a multi-period model, capital is likewise determined initially, but it also must be determined again at the beginning of each subsequent period. In order to optimize the amount of capital used, the capital-setting process will require a predetermined strategy. This strategy is dynamic: the subsequent capital

[6] For example, the state of California uses an investment pool for its domiciled insurers in liquidation. The pool contains only investment grade fixed income securities with duration less than 3 years (see California Liquidation Office 2014 Annual Report). New York is more conservative: funds are held in short-term mutual funds containing only U.S. Treasury or agency securities with maturities under 5 years (see New York Liquidation Bureau 2014 Annual Report).

amounts will depend on the values of the assets and of the insurer liabilities as they evolve. Even though the capital strategy is dynamic, there will be an optimal starting capital amount. Also, for each strategy, viewed at the beginning of the first period, there will be a distinct *expected* amount of capital at the beginning of each subsequent period.

3.4 Capital Funding

Since there is a cost to the insurer for holding capital, the insurer must be compensated for this cost. This cost is included in the premium. In a one-period model, the premium is paid up front and the loss is paid at the end of the period; there is no need to consider subsequent capital contributions. In a multi-period model, the liability estimate may increase over time, leaving the insurer's assets insufficient to adequately protect against insolvency. In such an event, the policyholders will be better off if the insurer's shareholders contribute additional capital. However, the insurer will be worse off due to the added capital cost. Nevertheless, if the premium includes the cost of additional capital funding, consistent with a particular funding strategy, it is economically practical for the insurer to make the capital contribution. Conversely, if the loss reserve decreases, it may be mutually beneficial for the insurer to remove some capital, consistent with the capital funding strategy.

For an ongoing insurer, there is a strong incentive to add capital as needed, since failure to do so may jeopardize the ability to acquire new business or renew existing policies. However, if technical insolvency occurs, it may not be feasible for the shareholders to add capital, since the prospect of a fair return on the capital may be dim. Thus, there are some limitations on capital additions. For a true runoff insurer, however, there is no incentive to add capital, so capital can only be withdrawn (which may occur if allowed by regulators).

3.5 Capital Definition

In a multi-period model, the premium will include the expected frictional cost of capital for all future periods. However, at the end of the first period, only the first-period capital cost is expended for the multi-period model, and so the balance becomes an asset that is available to pay losses. This premium component thus can be considered as *policyholder-supplied* capital, since it increases the asset amount and serves to mitigate default in exactly the same way as the owner-supplied capital in the one-period model. Similarly, if the premium contains a provision for the insurer's cost of bearing risk (a risk margin), that amount will also function as capital. Section 4.4 discusses the distinction between ownership capital and policyholder-supplied capital. Section 8.3 develops optimal capital with a risk margin.

4. BASIC MULTI-PERIOD MODEL

This section extends the one-period model to N periods and discusses some important differences between the two cases. The basic model developed here is designed to contain a minimal set of features that directly illustrates the optimal capital calculation. Other features, which may be necessary for practical applications, are discussed in sections 5 through 8.

The basic multi-period model follows a specific cohort of policies insuring losses that occur at the start of the first period and which are settled at the end of the Nth period. The model assumes that the insurer is ongoing, so that other similar policies are added at the beginning of the other periods. The basic model does not track these other policies; however, the prospect of profit from the additional insurance provides an incentive to add more capital to support the basic model cohort, if necessary.

4.1. Model Description and Assumptions

I start by adopting the basic assumptions of the one-period model, as developed in EBRM, and modifying some of them to fit the requirements of the multi-period model, as indicated below.

(1) Policyholders are risk averse with homogeneous risk preferences and their losses have the same probability distribution. Thus, the certainty-equivalent values of losses and default amounts are identical for each policyholder.

(2) There are no expenses (administrative costs, commissions, etc.). The only relevant costs are the frictional capital costs and the losses. These costs determine the premium.

(3) The cash flows for premium and the initial capital contribution occur at the beginning of the first period. The frictional capital cost is expended at the end of each period (before the loss is paid).[7] The entire loss is paid at the end of the *last* period. Other capital contributions or withdrawals may occur at the beginning of each subsequent period, depending on the insurer's capital strategy.

(4) The interest rate is zero. This simplification makes the exposition less cluttered (since the nominal values equal present values) and does not affect the key results. Section 8.2 provides results with a positive interest rate.

(5) Losses have no correlation with economic factors and consequently have no risk margin. Thus, since the investment return is also zero, the expected return on owner-supplied capital is also zero.[8] Section 8.3 analyzes results with a risk margin.

(6) The frictional capital cost rate is $z \geq 0$. It applies to the ownership capital defined in section 4.4.

(7) There is no cost to raising external capital (section 5.4 develops results that include this cost).

(8) There is no guaranty fund or other secondary source of default protection for policyholders. The only insolvency protection for policyholders is the assets held by the insurer.

(9) Capital adequacy is assessed only at the end of the period for regulatory purposes. Thus, an insolvency can only occur at the end of a period.

Additionally, we require some assumptions specific to the multi-period case that do not apply to a one-period model:

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[7] I chose this assumption to be consistent with the one-period model in EBRM. For the one-period model, this assumption avoids the issue of policyholder-supplied capital vs. ownership capital. If the loss is paid *before* the capital cost is expended, the optimal capital is determined from $\hat{Q}(A) = z \,/\, (1 + z)$, instead of $\hat{Q}(A) = z$, which is a simpler result that gives approximately the same optimal capital.

[8] This is a standard financial economics assumption; with no systematic risk, the required return equals the risk-free rate (which is zero here). There will be a positive expected return if a risk margin (discussed in section 8.3) is included.

(1) The ultimate loss is not necessarily known when the policy is issued, but is definitely known at the end of the Nth period (or sooner). This situation requires an intermediate estimate (the reserve amount) of the ultimate loss at each prior period. The reserve value is unbiased: it equals the expected value of the ultimate loss.

(2) The premium includes the *expected* FCC, since under a dynamic capital strategy, the capital amounts in future periods will depend on the random loss valuation and thus are also random.

(3) A capital strategy is used, wherein for each possible pair of loss and asset values at the end of each period, the insurer will add or withdraw a predetermined amount of capital.

(4) The policy term is one period. Section 7.4 discusses the case where the term is longer than a single period.

Since the certainty-equivalent value of losses and related expected default amounts are assessed from the perspective of each individual homogeneous policyholder, we scale the insurer model to portray each policyholder's *share* of the results. Therefore, it is useful to consider the model as representing an insurer with only a *single* policyholder.

In the multi-period model with N periods, variables that have a time element are generally indexed by a subscript denoting a particular period as time moves forward. The index begins at 1 for the first period and ends at N for the last period. Balance sheet quantities such as assets and capital are valued at either the beginning or end of the period, depending on the context. For example, C_1 represents capital at the beginning of the first period and A_1 denotes the assets for the first period after the capital cost is expended. For simplicity, I drop the subscript for the first period where the situation permits.

When developing optimal capital with backward induction (section 6) the index represents the number of *remaining* periods: e.g., C_3 denotes the initial ownership capital for a three-period model.

Optimal values are represented by an asterisk (e.g., C^*), certainty-equivalent quantities by a carat (e.g., $\hat{D}$), market values (used in risk margins) by a bar (e.g., $\overline{L}$) and random values by a tilde (e.g., $\tilde{C}$).

Note that under this simplified model, it is not necessary to distinguish between *underwriting* risk (the risk arising from losses on premiums yet unearned) and *reserve* risk (the risk arising from development of losses already incurred from prior-written premiums).

4.2 Stochastic Process for Losses

To analyze capital requirements, it is useful to categorize property-casualty losses into two idealized types, which are approximate versions of real-world processes. The first loss type is *short-duration*, e.g., property, where losses are settled at the end of the same period as incurred; a loss has at most a one-period lag between its estimated value when incurred and when ultimately settled. The second type is *long-duration*, e.g., liability coverage, where the lag is at least one period; if a loss occurs in a particular period, its value in a subsequent period will depend on its value in the earlier period.

For analyzing capital under the section 4.1 basic model, short-duration losses are one period, since the loss value cannot carry over to a subsequent period. Also, the expected value of losses in a subsequent period is independent of losses occurring in an earlier period. Since the per-policy mean loss (adjusted for inflation) does not change much over time, property losses generally follow a *stationary* stochastic process. With short-duration losses under the basic model considered to be one-period,[9] determining optimal capital is straightforward (see section 2), and so I turn to liability losses.

4.21 Long-Duration Loss Stochastic Process

Under a one-period model, the expected loss is *L*, which is a component of the premium. With a

[9] An exception is where the policy term is more than one period. This case is discussed in section 7.4.

multi-period model, we use the same notation for the initial loss estimate. However, there will be intermediate reserve estimates $\{L_1, L_2, \cdots, L_{N-1}\}$ at the end of the periods 1 through $N-1$. The realized value of the ultimate loss is denoted by L_N. Because we have assumed that the reserve estimates are unbiased, each reserve value L_t is the *mean* of the possible values for the next reserve estimate L_{t+1}. In other words, the difference $X_{t+1} = L_{t+1} - L_t$, or the *reserve increment*, has a zero mean. The sequence of reserve estimates is a *random walk*, which is a type of Markov process.[10] In a Markov process the future evolution of the value of a variable does not depend on the history of the prior values. In other words, conditional on the present reserve value, its future and past are independent. There cannot be a correlation between successive reserve amounts if the estimates are unbiased. The normal loss model in section 5.3 is an example of this stochastic process, which is an *additive* model since the increments are summed to determine successive values.

An alternative stochastic process that may characterize loss evolution is a *multiplicative* model. Here we define $Y_{t+1} = L_{t+1} / L_t$, which has a mean of 1 for all *t*. The *product* of the multiplicative random Y_t factors and the initial loss estimate L will give the ultimate loss value L_N. The lognormal loss model in section 5.3 is an example of this stochastic process. Notice that $\ln(Y_{t+1}) = \ln(L_{t+1}) - \ln(L_t)$, which is an additive random walk with a zero mean as described above.

For simplicity, I assume that the X_t values have the same type of probability distribution (e.g., normal) for all time values *t*. I also assume that the *variance* of X_t (denoted by σ^2) is constant per

[10] See Bharucha-Reid[1960].

period. In practice, this assumption may need to be modified.[11] Finally, I assume a similar regularity for the multiplicative model.

Notice that the variance of the ultimate loss L_N is the sum of the variances of the X_t sequence, or $N\sigma^2$. There is no covariance between any of the reserve increments due to the memory-less property of the Markov process (a non-zero correlation would imply that the prior reserve history could help predict the future reserve values). The X_t variance exists because the flow of information (positive and negative) regarding the ultimate loss value is random. The subsequent estimates of ultimate value are determined by information that becomes revealed over time, such as how many claims have occurred, the nature of the claims, the legal environment, inflation and so forth.

4.22 Certainty-Equivalent Stochastic Process

The certainty-equivalent loss values will evolve according to a stochastic process parallel to that of the underlying losses. Generally, if the policyholder risk aversion is based on utility theory, the risk value embedded in the CE losses is approximately proportional[12] to the loss variance. The relationship is exact if the loss values are normally distributed and policyholder risk aversion is represented by *exponential utility*. For this additive stochastic process with a constant[13] per-period loss volatility, the CE expected loss at the end of N periods is then

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[11] This assumption can be modified to provide a specific variance for each period, as will be necessary for practical applications. The actual distribution may vary according to the elapsed claim duration. For example, the long discovery (with claims incurred but not reported) phase for high-deductible claims will imply a low variance for the reserve estimates for the first few years. Scant information regarding the claims arrives over this time span, so there is little basis to revise the initial reserve.

[12] See Panjer et al. [1988], page 137.

[13] If the loss volatility is not constant, then the term $N\sigma^2$ is replaced by $\sigma_1^2 + \sigma_2^2 + \cdots \sigma_N^2$, where σ_i^2 is the variance of the *i*th period loss volatility.

$$\hat{L}_N = L + aN\sigma^2 / 2, \tag{4.221}$$

where *a* is a constant that indicates the degree of risk aversion. Therefore, the CE expected loss increases each period by the risk value $a\sigma^2 / 2$. Since the CE loss mean increases linearly with the time horizon, we can create a parallel CE stochastic process by satisfying equation 4.221. Appendix B shows how the *N*-period CE distribution of losses or assets is determined under the normal-exponential model and includes a numerical example.

Notice that equation 4.221 represents the CE expected loss value with *N* periods remaining; as the loss evolves there will be fewer periods left and the risk value will diminish (it will be zero when the loss is settled).

Appendix A illustrates a two-period stochastic process with a simple numerical example using a discrete probability distribution.

4.3 Premium and Balance Sheet Model

Following the one-period model, the premium for the multi-period case equals *L* plus the expected capital cost. However, the capital for each period after the initial period will be determined by the evolving loss estimate, so it also will be a random variable. Consequently, the capital cost component of the premium will be the *expected value* of the sequence of capital costs. Let *C* denote the *ownership capital*, which is the amount of capital contributed initially (here I drop the subscript 1 for the first period). For a specific capital funding approach, under an *N*-period model, let $\tilde{C}_i$ be the capital amount at the beginning of the *i*th period.

Assume that the frictional capital cost is proportional to the *ownership capital* (see section 4.4 for a discussion of capital sources) at the rate *z*. As shown in EBRM, the double-taxation component of the capital cost depends solely on the ownership capital amount.[14] The expected capital cost for all periods is then $K = zC + E[\tilde{C}_2 + \tilde{C}_3 + \cdots + \tilde{C}_N]$. Accordingly, the fair premium equals $\pi = L + K$, which has the same form as in the one-period case.

The expected value of the future capital amount or of the capital cost should be calculated using *unadjusted* probabilities, since, like the expected return on capital, the frictional capital cost rate z does not depend on policyholder risk preferences. Also, the insurer is already compensated for the risk it bears through the risk margin built into the premium. Although the risk margin is zero here in the basic model, a more general model, such as in section 8.3, will include it.

This premium model forms the basis for pricing methods that use the present value of expected future costs and whose losses have embedded risk margins (see sections 8.2 and 8.3). The present value of the expected capital costs is determined by discounting them at a risk-free rate.

When the policies are written, the initial assets equal the owner-contributed capital plus the premium, or $C + \pi = C + L + K$. With a zero interest rate, these assets are cash in the basic model. The liabilities are the expected losses, the expected capital cost[15] and the ownership capital, which is the residual of assets minus the obligations to other parties. At the end of the first period, before the loss is paid, the capital cost for that period is expended, leaving the amount of assets available to pay losses, denoted by A, as

$$A = C + L + K - zC \, . \tag{4.31}$$

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[14] Other frictional capital cost components might depend on total capital or total assets, but since they are likely to be smaller than the double-taxation amount, I have assumed that they also are proportional to the ownership capital.
[15] As discussed in EBRM, this amount is primarily an income tax liability.

4.4 Ownership Capital and Total Capital

For the basic one-period model, the capital definition is straightforward. At the beginning of the period, the insurer's owners supply a capital amount C, and the policyholders supply the premium, equal to $L + zC$. Since the capital cost amount zC is expended before the loss is paid, the amount of assets available to pay the loss is $A = L + C$.

For a multi-period model, however, the amount available to pay losses after the first period is *greater* than $L + C$ by the amount $K - zC > 0$, which represents the expected capital cost for the remaining periods. Since this additional amount reduces default in exactly the same way as the owner-supplied capital, it may be considered as *policyholder-supplied* capital. Therefore it is useful to define the *total capital* as the available assets minus the expected loss, which for the basic multi-period model is

$$T = C + K - zC \; . \tag{4.41}$$

Notice that for a one-period model, we have $T = C$, and for two or more periods, $T > C$.

It is important that the ownership capital measurement be consistent with the premium determination. Here I use *fair-value* (also known as mark-to market) accounting, where the value of obligations is the amount they would be worth in a fair market exchange[16] and are thus equal to the fair premium. From equation 4.41 it is simple to determine the fair-value capital from the total capital and vice-versa. For brevity, I use OC to denote ownership capital.

With a risk margin, discussed in section 8.3, we have a similar situation: the risk margin

[16] An important property of fair value accounting is that, if the product is fairly priced (so that its components are priced at market values), there is no profit generated when the product or service is sold. Instead, the profit is earned smoothly over time as the firm's costs of production or service provision are incurred. For an insurer, this means that the profit will emerge as the risk of loss is borne.

compensates the insurer for bearing risk and is a premium component in addition to the expected loss. Like the unexpended expected capital cost, it provides additional default protection. However, in fair-value accounting, the risk margin is not considered as ownership capital.

For the subsequent sections, I present most results using *the total capital* definition. Where appropriate, I show the OC for comparison.

4.5 Capital Funding Strategies

In order to determine the expected capital cost, we need to know how much capital will be used for each period. As discussed in section 3.4, the amount will depend on the loss amount at the end of the prior period: if the amount is large, it may be necessary to add capital; if the amount is small enough, capital might be withdrawn. Define a *capital funding strategy* as a set of rules that assigns a specific amount of capital, called the *target* amount, to the beginning of each period, corresponding to each possible loss value at the end of the prior period. Note that there is not necessarily a *unique* capital amount for each loss amount, since a range of losses can produce a single capital amount (such as region 2b in section 5.42).

There are several basic capital funding strategies that an insurer might use. I describe the most relevant ones below, starting from the least to the most dynamic method.

Fixed Assets (FA): under this approach, the insurer's owners supply an initial capital amount, with no subsequent capital flows until the losses are fully settled. Thus, the initial assets remain constant until the losses are paid. The capital amount will vary over time, since loss estimates will fluctuate and the capital equals assets minus liabilities. This method is used in Lowe et al. [2011] to determine capital for a runoff capital model. Although it is viable for a true runoff insurer, it will not be for an ongoing insurer, whose capital level generally responds to the level of its loss liabilities. For example, if an insurer's losses develop favorably, causing its capital amount to increase above a target level dictated by the strategy, then the insurer will usually reduce its capital amount.

Capital Withdrawal Only (CW): with this strategy, capital is withdrawn if the asset level becomes high relative to the losses and therefore capital exceeds a particular target amount. A common method for withdrawing capital is through dividends to shareholders.[17] However, no capital is added if assets become lower than the target level. Except possibly for some mutual insurers, this method also does not represent actual practice, where, within limits, insurers will add capital if the existing capital amount is below the target level.

Add Capital if Solvent (AC): here, capital is withdrawn if a particular target level is reached, and capital is added if assets are below the solvency level. However, if the insurer becomes technically insolvent, then no capital is added. In this event, the insurer usually is taken over by a conservator. The incentive for shareholders to fund capital additions comes from the prospect of adding new business, which is difficult to accomplish without adequate capital. Note that a less restrictive threshold (where insurers are slightly insolvent) might be used in the event that shareholders consider the franchise value of the insurer to be valuable enough. However, the results of this assumption would be analytically similar to using a strict solvency/insolvency threshold. The main point here is that there is an upper limit to losses beyond which capital is no longer added.

A variation of the AC strategy, discussed in section 5.4, is where there is a cost to raising capital externally. I have labeled this strategy as ACR.

Full Recapitalization (FR): this approach is similar to AC, but the insurer, even if technically insolvent, will add sufficient capital to regain the target level. However, in order to provide an adequate incentive for the shareholders to provide capital if the insurer becomes insolvent, the

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[17] For a mutual insurer, the dividends will go to policyholders, who are the insurer's owners and therefore serve as shareholders. A mutual insurer's dividends can also be used as part of its pricing strategy.

policyholders must accept a cash settlement for their claims; the amount equals the asset value. The insurer (or a different insurer) then agrees to insure the loss liability again and the policyholders pay a new premium for the reinstated coverage. The insurer's owners then provide adequate capital for the insurance. This transaction, in effect, converts the technical insolvency into a cash or "hard" insolvency. Thus, it is possible for the insurer to default multiple times before the loss is settled. As discussed in section 5.2, the FR approach is theoretically superior to the other three methods in that it provides the highest consumer value for the insurance coverage. However, it is not feasible: normally, the policyholders will enter receivership rather than take back their liabilities and insure them again with a different insurer.[18]

Other strategies, such as only adding capital, are possible. However, I have included only the strategies that are used in practice or that illustrate important concepts.

Let $T^{*}_{t+1}(L_t)$ represent the target total capital amount at the beginning of period $t + 1$ given that the value of the loss at the end of period t is L_t. Thus the required assets at the beginning of period $t + 1$ are $L_t + T^{*}_{t+1}(L_t)$, and the indicated capital flow (i.e., addition or withdrawal) is the required assets minus the prior-period assets:

$$CF_t = L_t + T^{*}_{t+1}(L_t) - A_t. \tag{4.51}$$

The above four capital funding strategies, plus the ACR variant, can be characterized by the regions of L_t for which the indicated capital flow CF_t is permitted. The first region is $A_t < L_t$,

[18] One huge impediment to practically applying the FR method is that the insurer and the policyholders may have different opinions on the value of the loss reserve estimate. Another problem is that this capital funding method also requires either that policyholders *without claims* contribute enough to pay for their possible future incurred-but-not-reported (IBNR) claims or for the IBNR reserve to be divided among the existing claimants.

where the insurer is technically insolvent. The second is $[A_t - T^*_{t+1}(L_t)] < L_t < A_t$, where the insurer is solvent and capital can either remain the same[19] or increase if permitted. The third is $L_t < [A_t - T^*_{t+1}(L_t)]$, where capital is withdrawn if permitted. Notice that the regions do not depend on the expected loss amount; they depend only on the asset amount and the required capital amount for the second period. However, the expected loss (and the other distribution parameters) determine the respective probabilities that the loss falls in each of the three regions.

To illustrate, assume that the required total capital for the second period is 600 and is independent of the first-period loss value (i.e., it depends only on the variance, as under the normal distribution). The asset amount is 1400, which establishes the boundary between region 1 and region 2. If losses are less than 800, the remaining capital exceeds the required capital of 600 = 1400 – 800. Therefore, region 1 contains losses exceeding 1400, region 2 has losses between 800 and 1400 and region 3 contains losses less than 800. For region 1, capital is added only for FR. For region 2, capital is added for AC and FR. For region 3, capital is withdrawn for all funding strategies except FA.

Table 4.51 summarizes the capital flows permitted by the different capital strategies. A minus indicates a withdrawal, a plus represents an addition and a zero indicates that capital remains the same.

[19] Under the ACR strategy and the FR strategy with a capital-raising cost, there may be a sub-region of region 2, bordering on region 3, where capital remains the same. As shown in section 5.4, due to the cost of raising capital, it will be sub-optimal to add capital in this region, and also sub-optimal to withdraw it.

Table 4.51
Permissible Capital Flows
by Capital Strategy and Loss Range
Illustrative Example

Region	Loss Range	FA	CW	AC	FR
1	A > 1400	0	0	0	+
2	800≤A ≤ 1400	0	0	+	+
3	A ≤ 800	0	–	–	–

Each of these strategies may have a different expected capital cost and therefore the premium will depend on the strategy used. Notice that after the initial capital is established, the chosen strategy will produce a unique sequence of subsequent capital amounts corresponding to the sequence of actual loss estimates.

Since the insurer is fairly compensated up front for its capital costs, the capital suppliers (shareholders) will provide whatever capital amount (both for initial and subsequent periods) is desired by the policyholders. This also means that the investors are indifferent to the capital *strategy* desired by the policyholders, since the premium compensates the owners for the expected capital costs under the strategy. Therefore, for each capital strategy, we can determine the initial capital amount that maximizes the policyholder's consumer value. Then the strategy with highest consumer value (or the lowest solvency cost) is the optimal strategy and can be used to determine capital for similar types of insurance. A particular strategy is considered more *efficient* than another if it produces a higher consumer value.

4.6 Efficiency and Feasibility of Capital Funding Strategies

Assume a two-period model and that initial assets for each strategy are fixed at A_1. At the end of the first period, whatever the loss estimate L_1, there is a single period remaining. We already know how to find the optimal capital for one period. Defining the required total capital in section 4.5 as the optimal capital, the optimal capital for the beginning of period 2 is $T_2^*(L_1)$. Thus, if the actual capital T_1 exceeds $T_2^*(L_1)$, the additional capital cost (from carrying the capital into the second period) will be greater than the reduction in the CE expected default value for the second period (by definition of the optimal capital), so policyholders will gain by a capital withdrawal to attain optimal capital. Note that this situation occurs in region 3 of Table 4.51. Consequently, CW is a superior strategy to FA, which we can represent as CW > FA.

A similar argument shows that AC > CW. If the loss estimate is between initial assets minus $T_2^*(L_1)$ and initial assets (region 2), increasing capital will increase the capital cost less than it changes the CED value. In parallel fashion, we have FR > AC.

However, as discussed in section 4.5, FR is not feasible in practice. AC is feasible for most insurers and CW, although feasible, is less efficient than AC. So CW is not a good choice unless it is not possible to raise capital externally. Therefore, for most insurers, the most efficient feasible choice of the four strategies is AC. Accordingly, the subsequent sections in this paper primarily use the AC strategy. Nevertheless, it is informative to compare results between the different strategies. In particular, the FR strategy provides an important baseline, since it produces the highest consumer value and thus theoretically is the most efficient strategy. It also has the important feature that it converts a multi-period model into a *series of one-period models*.

Because of the single-period conversion property of the FR strategy, the required adjusted probability distributions can be analytically tractable, and it is relatively easy to calculate the optimal capital for the start of each period. This is usually not the case for the AC and CW strategies.

5. OPTIMAL TWO-PERIOD CAPITAL

In order to determine multi-period optimal capital, it is useful to begin by extending the one-period model to two periods. In the two-period exercise, we gain valuable insight regarding multi-period capital dynamics. The two-period results are readily extended to additional periods in section 6 using backward induction. The results here in section 5 use an example with a normal stochastic loss process. However, I also describe the general method to derive optimal capital for other stochastic processes.

First, I address the simple case where there is no cost to raising capital from external sources. Then, in section 5.4, I introduce a cost of raising capital and show how this changes the AC optimal capital.

5.1 Expected Default with the AC Strategy

An important constraint in modeling capital for multi-period losses is that a technical insolvency normally forces an insurer into conservatorship. This event means that losses will continue to develop while assets remain fixed until the losses are settled. Here I assume that the insurer enters conservatorship *immediately* when the technical insolvency occurs at the end of a particular period.

Conservatorship adds another dimension to the CE expected default calculation that is absent for a one-period model. From section 2, the CED for a one-period loss is denoted by $\hat{D}$. Define $\hat{G}$ as the unconditional ultimate CED for an insurer entering technical insolvency at the end of the first period. For a discrete loss process let x_i for $i = 1, \cdots, n$ denote each possible value of the first-period loss L_1 that exceeds initial assets. Let $\hat{p}(x_i)$ represent the certainty-equivalent probability that x_i occurs and $\hat{D}_2(x_i)$ the CE expected second-period default given x_i. The CE expected default due to a technical insolvency is therefore

$$\hat{G} = \hat{p}_1(x_1)\hat{D}_2(x_1) + \hat{p}_2(x_2)\hat{D}_2(x_2) + \cdots + \hat{p}_n(x_n)\hat{D}_2(x_n). \qquad (5.11)$$

To illustrate this, I approximate a normal stochastic loss process using a discrete probability distribution for the independent loss increments. This numerical example is shown in Appendix A. The value of $\hat{G}$ is 0.9029, which exceeds the first-period $\hat{D}$ value of 0.3144.

Observe that $\hat{G}$ depends on the variance of loss development *beyond* the first period (i.e., the ultimate variance), while $\hat{D}$ only depends on volatility *during* the first period. For a positive second-period variance, the mathematical properties of the default calculation ensure that $\hat{G}$ is *greater* than that of the original first-period default: $\hat{G}$ cannot be negative; it equals zero if the loss develops favorably. This asymmetry increases the expected ultimate default amount beyond its initial first-period value regardless of the first-period loss amount.

5.2 Optimal Two-period AC Capital

A particular value of initial capital *C* will establish the assets *A* available to pay the loss at the end of the first period (equation 4.31). This asset amount will thus uniquely determine the CE expected default $\hat{G}$ for the first period, as discussed in section 5.1. The amount *A* will also uniquely determine the CED for the second period since the capital strategy is predetermined. The total CE expected default for the insurer is the sum of the CED values for the first and second periods.

For a continuous distribution of losses, with *x* denoting the first period loss value, the equivalent of equation 5.11 is

$$\hat{G} = \int_A^{\infty} \hat{p}(x)\hat{D}_2(x)dx\,. \tag{5.21}$$

If the insurer remains solvent at the end of the first period, there is one period remaining: it can become insolvent at the end of the second period. However, from section 2, for each loss value there is an optimal amount of capital and a corresponding optimal CED amount, represented by

$\hat{D}^*(x)$. The insurer will add or withdraw capital to reach the optimal beginning second-period capital. The CE expected default in the second period is then

$$\hat{H} = \int_0^A \hat{p}(x)\hat{D}^*(x)\,dx\,. \tag{5.22}$$

In words, $\hat{H}$, the CED for the second period, is the sum of the optimal one-period CED for each first-period loss value less than the asset amount, weighted by the CE probability of the loss value. Observe that the limits of integration span loss amounts from 0 to A, while the limits for $\hat{G}$ span amounts greater than A. Consequently, the insurer's total CE expected default for both periods is $\hat{G} + \hat{H}$.

From section 4.3, the premium for a multi-period loss coverage is $\pi = L + K$, where *K* is the expected capital cost for all periods. For two periods, the expected amount of ownership capital used is the initial first-period OC (a fixed amount) plus the expected second-period initial OC (a random amount determined by the first-period loss). Let $C_2^*(x)$ be the optimal second-period initial OC given that $L_1 = x$. Under the AC strategy the second-period initial OC is the optimal OC for a one-period insurer with expected loss L_1. Therefore we have

$$K = zC + z\int_0^\infty p(x)C_2^*(x)dx\,. \tag{5.23}$$

Here *p*(*x*) is the *unadjusted* probability of loss, since we have assumed that the insurer will incorporate the actual expected amount of capital into the premium. For simplicity, rather than using the asset value A, I have used an infinite upper limit.[20]

The consumer value of the insurance transaction is $V = \hat{L} - \pi - \hat{G} - \hat{H}$. The optimal initial available asset value is found by maximizing V, or alternatively, minimizing the *solvency cost*

$$S = \hat{G} + \hat{H} + K \,. \tag{5.24}$$

Because $\hat{G}$ is not analytically tractable for important probability distributions such as the normal, we need to use numerical approximation methods to find the optimal assets in these cases. Once the optimal assets are found, we use equation 4.31 to determine the optimal capital. Section 5.3 outlines an approach for the normal and lognormal stochastic processes.

For the FR strategy, the insurer is recapitalized at the end of the first period to the optimal second-period amount. So, viewed from the beginning of the first period, the solvency cost for the second period is the optimal amount for that period as if we had just begun that period. Therefore, the initial capital for the first period is independent of the second-period loss distribution, and depends only on the potential loss values for the first period.

Section 5.1 showed that, for a given initial asset level, the CE expected default for the AC strategy is higher than that for the FR strategy. This implies that the optimal initial total capital for the AC strategy is *higher* than for the FR strategy, which is the theoretically most efficient strategy. This result is reflected in the section 5.3 numerical examples with the normal stochastic loss process.

To prove this result, assume that we use an AC strategy, but the initial total capital is the optimal total capital for an FR strategy. The AC certainty-equivalent default $\hat{G}$ is greater than the optimal CED under FR. Also, the derivative $\partial \hat{G} \,/\, \partial A$ is a weighted average of the $\hat{Q}$ values for losses

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[20] The error in this approximation will be small if the default probability is small. In the section 5.3 example, the difference in optimal capital is 333.34 – 333.15 = 0.19, an error of 0.06%.

greater than A. Each of the component $\hat{Q}$ values in the weighted average is higher than z, so adding capital at the margin will reduce $\hat{G}$ more than it will increase the capital cost. Consequently, the optimal AC total capital will be greater than the optimal FR total capital[21] for two periods and the optimal AC solvency cost will be greater as well.

5.3 Optimal Two-period Capital for Normal Stochastic Processes

In this section I use the normal stochastic process from section 4.2 to calculate numerical results. Here the period-ending loss distribution is normal. This distribution is continuous, and serves to illustrate dynamic loss development. The policyholder risk aversion is based on exponential utility; thus optimal capital can be determined from the resulting CE values, as shown in Appendix B. The numerical example developed here is expanded in subsequent sections to demonstrate results for variations of the basic model. These results are intended to elucidate the general method for determining optimal capital; a practical application will likely involve more complex modeling. Although the *lognormal* loss process is perhaps better suited to modeling insurance loss development,[22] I have chosen to use the normal model, which is simpler to explain and which provides tractable results for a joint loss and asset distribution (see section 9). Under the lognormal process, the conditional one-period optimal capital and CED are *proportional* to the expected loss, while under the normal distribution, these values are independent of the expected loss. The results for a lognormal loss process are similar, [23] however.

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[21] Since the premium contains the expected capital cost for both periods, the optimal first-period FR ownership capital equals the optimal OC for a one-period model, less the expected capital cost for the second period. Essentially, in this case, compared to the one-period model, the policyholder has prepaid the second-period capital cost, so the optimal initial ownership capital is *less than* in the one-period model by the amount of the prepayment.

[22] The lognormal distribution has been used by several authors (see Wacek [2007] and Han and Goa [2008]) to analyze the variability of loss reserves.

[23] For the same periodic loss volatility, the optimal capital for the lognormal process is slightly higher than that for the normal counterpart.

With a normal loss process, the optimal capital and CED for one period are constants independent of the expected loss (but are a function of the standard deviation). This property facilitates the calculation of optimal capital for two or more periods. Appendix B develops a numerical example to illustrate optimal capital under the normal stochastic loss process with exponential utility, which is labeled as the normal-exponential model. I extend the example to illustrate results in subsequent sections of the paper.

The example uses a two-period normal stochastic loss process with a mean of 1000 and variance of the loss increment equal to 100^2 for each period. The CE value of the expected loss after one period is 1050 and the risk value (the CE of the loss minus its expected value) at each development stage is strictly proportional to the cumulative variance as in section 4.22. Thus, the CE value of the ultimate loss at the end of the second period is 1100.

The frictional capital cost is $z = 2\%$. The optimal one-period total capital is 291.62 and the optimal two-period initial total capital is 333.34.

Table 5.31 summarizes the optimal AC results. Here I compare the optimal two-period AC strategy with that of the optimal FR strategy. The table also shows results for the AC strategy using the optimal FR initial total capital as the initial capital for the AC strategy.

Table 5.31
Optimal AC and FR Strategy Comparison
Normal-Exponential Example
(Source: Appendix B.3)

Strategy	Initial Total Capital	1st Period CE Default Probability	1st Period CED	2nd Period CED	1st Period Capital Cost	2nd Period Capital Cost	Solvency Cost
FR Optimal	291.62	0.0200	0.7852	0.7852	5.7158	5.8325	13.1187
AC Using FR Capital	291.62	0.0200	2.1325	0.7695	5.7158	5.8325	14.4503
AC Optimal	333.34	0.0073	0.7514	0.7794	6.5502	5.8325	13.9136

Notice that the optimal solvency cost for the AC strategy has a lower total CED for both periods (1.5309) than does the FR strategy (1.5704). However, the AC capital cost is larger, giving a higher AC optimal solvency cost.

5.4 Two-Period AC Model with Cost of Raising Capital

The earnings for an ongoing insurer are usually positive; these provide internally generated capital which normally is sufficient to maintain its operations. Thus, most of the time it will withdraw capital (usually as distributions to owners) to maintain the desired capital level. If earnings are negative, it may be necessary to raise ownership capital externally, through issuance of bonds or equity capital. The initial basic model of section 4 assumed that the cost of raising capital externally is zero. This is not realistic, since it is generally considered that there is a positive cost of raising external capital for businesses (see Myers and Majluf [1984]), including insurers (see Harrington and Niehaus [2002]).

A portion of this cost is due to the administrative expense of the capital issuance, such as investment bank fees. The other part of the cost is due to *signaling*, where if an insurer needs additional capital due to low earnings, investors may believe that the management is poor. Thus, the capital suppliers will require a high return on the capital provided and the value of the company to existing shareholders will be diluted. This effect is especially prominent when most other insurers do not require additional capital.[24]

5.41 Linear Model for Cost of Raising Capital

To model the cost of raising capital (abbreviated by CRC), assume that the cost is a rate *w* times the amount of capital raised.[25] We continue to assume that no capital is raised if the insurer is technically insolvent. Also assume that the insurer is already in business, so that its first-period capital is not raised externally.[26]

For a two-period model, at the end of the first period, there is one period remaining. If it is not necessary to raise capital, the optimal capital for the beginning of the second period is determined by equation 2.23. However, if capital is raised at that point, there is an additional capital cost *w* to the insurer beyond *z*, the cost of holding capital.

Let C_R represent the initial second-period ownership capital after having raised capital and C_E the ending first period OC. Thus, the amount of capital raised is $C_R - C_E$. We need to distinguish between the optimal amount of capital given that it is raised externally and the amount if it is generated internally. Hence the distinct notation for the capital amount given that it is raised externally. The total capital cost in the second period is then $zC_R + w(C_R - C_E)$.

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[24] In the event of an *industry-wide* catastrophe or pricing cycle downturn, the signaling effect may not be significant. In fact, the prospect of near-term increased insurance prices can spur investment in the property-casualty industry.
[25] An alternative formulation is to assume that the cost of raising capital increases as the insurer nears insolvency, but this will be more difficult to model.
[26] The one-period model in EBRM implicitly assumed that there was no cost of raising capital. A solvent ongoing insurer with one-period losses will need to raise capital (to the optimal level for the next group of policyholders) if the ending loss amount is large enough. This effect will change the optimal initial capital slightly.

Because the marginal amount of capital raised carries a cost of $z + w$, following the section 2.2 analysis, the optimal second period capital, given that it is externally raised, is determined by

$$\hat{Q}(A_2) = z + w, \tag{5.411}$$

where A_2 is the second-period available assets. Since we assume that w is positive, the optimal second-period capital with a CRC is *less* than that if there were no CRC: since new capital raised is expensive with a CRC, the insurer will use less of it; the policyholder is satisfied, having achieved the optimal balance of price and security. Denote the optimal second-period OC, given that capital is raised externally at the end of the first period, by C_R^*.

5.42 Optimal Two-Period Capital with CRC

With a positive cost of raising capital, we can modify the AC strategy to produce an optimal initial capital amount. To distinguish an AC capital strategy with a positive cost of raising capital from one with a zero cost, I abbreviate the CRC version to ACR.

Under the dynamic ACR strategy, the CRC is incurred if the first-period loss estimate L_1 is such that the ending first-period OC is between zero and C_R^*. Thus, the capital flows depend on *four* distinct regions based on the first-period ending OC amount C_E. To illustrate this, I expand Table 4.51 by splitting region 2 into two sub-regions. Table 5.421 shows the capital flows by region:

Table 5.421
Capital Flows by Region
Two-Period AC Strategy with Cost of Raising Capital

Region	Capital Carried Forward	Capital Raised	Capital Withdrawn
1: $C_E < 0$	0	0	0
2a: $0 < C_E < C_R^*$	C_E	$C_R^* - C_E$	0
2b: $C_R^* < C_E < C^*$	C_E	0	0
3: $C^* < C_E$	C^*	0	$C_E - C^*$

In region 1, the insurer is technically insolvent, so there are no capital flows. In region 2a, the ending OC is lower than the optimal capital needed if raising capital, so the capital amount C_E is carried forward and capital is added to reach C_R^*. In region 2b, the amount C_E is carried forward, but the ending capital is greater than C_R^*, so no capital is raised. The ending capital is also lower than C^*, so none can be withdrawn either. In region 3 the ending capital is more than C^*, so the excess is withdrawn.

Denote the region 2a expected amount of capital carried forward by EF_a. We have

$EF_a = \int_{A-C_R^*}^{A} (A-x)p(x)\,dx$, where x is the ending first-period loss value and $p(x)$ is the unadjusted probability of x occurring. This integral equals

$$EF_a = D(A-C_R^*) - D(A) - C_R^* Q(A-C_R^*), \qquad (5.421)$$

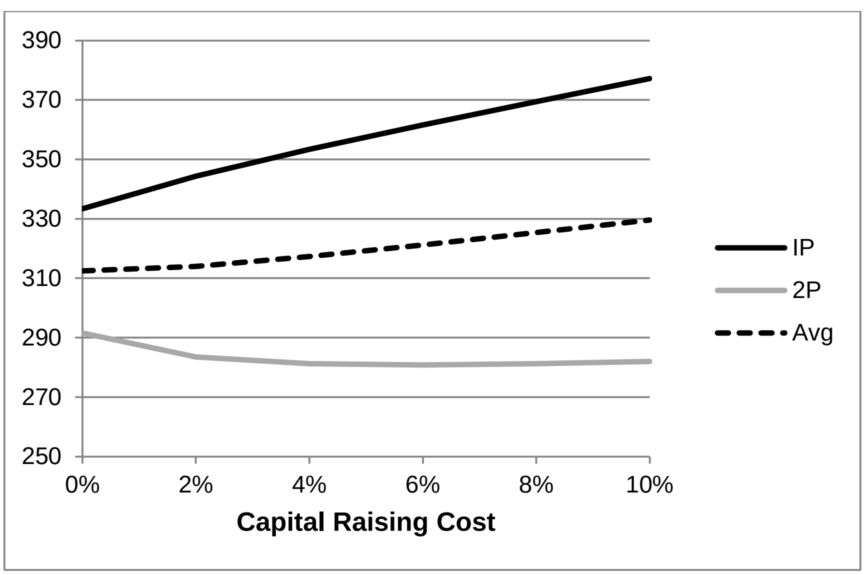
where the expected default and the default probability values are determined by unadjusted probabilities. The expected amount of capital carried forward for region 2b is developed in a similar fashion, and equals

$$EF_b = D(A - C^*) - D(A - C_R^*) - C^* Q(A - C^*) + C_R^* Q(A - C_R^*). \qquad (5.422)$$

Equations 5.421 and 5.422 allow one to calculate the expected capital cost for the second period under the ACR strategy. The expected CE default amounts are determined by the optimal values associated with C_R^* and C^*, so we can determine the optimal solvency cost and optimal capital. Appendix C illustrates this calculation by extending the section 5.3 normal example.

Figure 5.421 compares the optimal initial, expected second-period and average total capital obtained by varying *w* in this example from 0 to 10%.

Figure 5.421
Optimal Initial Total Capital Amount by Cost of Raising Capital
Two-Period AC Strategy
Normal-Exponential Example



Notice that increasing the CRC raises the initial first-period optimal total capital, with the expected second-period optimal capital being lower than that for the optimal one-period capital. The second-period capital is diminished because the second-period capital cost goes up due to the CRC and the insurer (on behalf of the policyholder) will use less of it. The initial capital increases because the insurer will avoid some of the high second-period cost by having a higher initial capital and carrying more of it into the second period. Also notice that increasing the CRC also raises the average amount of capital over both periods.

The capital raising cost will vary by insurer, and is likely to be lower for established insurers with better access to the capital markets. Thus, the CRC is another variable to consider when assessing risk-based capital.[27]

5.5 Insurers with Limited Ability to Raise Capital

Besides depending on the cost of raising capital, the optimal capital amount also depends on the *ability* of insurers to raise capital. It is well-known that the organizational form of insurers dictates how they may raise capital (see Harrington and Niehaus [2002] and Cummins and Danzon [1997]). In particular, depending on the details of their structure, mutual insurers may have difficulty raising capital externally.[28] In the case where an insurer cannot raise external capital, the best capital strategy is capital withdrawal (CW). Note however, that this strategy will represent an upper limit to optimal capital for a mutual insurer, since the insurer can raise additional capital internally by charging its policyholders a higher premium.[29]

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[27] With a CRC, even under a FR strategy the optimal initial capital will be larger than without the CRC, since capital must be stockpiled early to avoid the cost of subsequently raising it. Thus, for the FR strategy with a CRC the initial capital depends on the volatility of *future losses*, not just the behavior of current period losses.

[28] Some mutual insurers have issued *surplus notes*, which are similar to equity in terms of capital structure, but are a type of risky bond to investors. According to A.M. Best [2003], the major issuers of surplus notes were usually large insurers with more access to capital markets, while small or mid-size insurers could only issue surplus notes in limited amounts with short maturity.

[29] However, this method is limited since the policyholders will tend to migrate to other insurers if the premium is too high.

Under CW, all capital flows (except for the initial capitalization) are withdrawals; capital increases arise from positive earnings. Using the section 5.4 example, the optimal initial CW total capital amount is 433.61, with an expected second-period optimal capital of 291.62 and average over the two periods of 362.62. The solvency cost of this optimum position is 16.29. For comparison, the solvency cost of the section 5.42 AC strategy with a 4% CRC is 14.76.

Harrington and Niehaus show that mutual insurers carry more capital than stock insurers having the same risk. This result supports the analysis presented here.

Although the solvency cost (and hence the consumer value) for the mutual insurer is inferior to that of the section 5.4 insurer, the policyholder is *not necessarily worse off*. A mutual policyholder is also an owner of the insurer and receives dividends if the mutual is profitable. These distributions are not taxable at the personal income level. However, a similar policyholder of a stock insurer with an equivalent stake in that insurer would be subject to income taxes on the capital distributions. This tax-free benefit increases the consumer value of the mutual insurance purchase. To illustrate, suppose that the personal income tax rate on the capital distributions is 20% and the expected return on capital is 8%. The average ownership capital for the section 5.4 stock insurer (with w = 4%) is 317.30. Thus, the expected return to the policy/equity holder is 25.38. The tax on this amount is 5.08 = 0.20(25.38). The stock policyholder's consumer value after the personal income tax is the risk value minus the solvency cost minus the income tax: 80.16 = 100.00 – 14.76 – 5.08. The mutual policyholder's consumer value, with no personal income tax, is in fact higher: 83.71 = 100.00 – 16.29.

To the extent that regulatory capital requirements are related to the optimal capital that insurers might carry, then the analysis here suggests that risk-based capital should be *higher for mutual insurers* than for stock insurers having the same default risk.

6. OPTIMAL CAPITAL FOR MORE THAN TWO PERIODS

This section determines optimal capital for multiple periods by extending the two-period model for the capital strategies using the backward induction method. Here I outline the method generally and apply it to the AC and ACR strategies.

6.1 General Backward Induction Method

The backward induction method determines a sequence of optimal actions or results by starting from the end of a problem with discrete stages and working backwards in time, to the beginning of the problem. It uses the output of each prior stage to determine an optimal action based on the information available for the particular stage. This course proceeds backwards until one has determined the best action for every possible situation at every point in time. Backward induction is used extensively in dynamic programming and game theory.[30]

To apply backward induction for a capital strategy where there is no CRC, define an index i for each stage, where i is the number of periods remaining until the ultimate loss is determined. At each stage i, we use three optimal quantities that have been determined from the prior stage, and may depend on the loss value x from stage i: the optimal ownership capital $C_{i-1}^{*}(x)$, the optimal CED $\hat{D}_{i-1}^{*}(x)$ and the optimal capital cost $K_{i-1}^{*}(x)$.

For stage i, we start with the optimal asset amount from the prior stage $i-1$ and calculate the solvency cost. We vary the asset amount until the optimal solvency cost is attained, and record the values of the above three optimal quantities. The process is repeated until the Nth stage is complete. The result is the optimal initial capital, CED and capital cost for an N-period model. The intermediate stage results will give the optimal quantities for all models of lesser duration that have

[30] For example, see Von Neumann and Morgenstern [1944].

the same sequence of loss increment variances per period.[31] Accordingly, for a model with constant volatility per period, we will get the optimal results for all models with N or fewer periods.

6.2 Backward Induction Method with AC Strategy

Under the AC capital strategy for stage *i*, the solvency cost has four components: (1) the CED for technical insolvency in the stage, (2) the expected CED for future insolvency, (3) the capital cost for the stage and (4) the expected future capital costs. The first two components represent the total CED for all periods through stage *i*, denoted by $\hat{D}_i$, and the last two represent K_i, the total capital cost for all periods. Therefore we can represent the solvency cost as

$S_i = \hat{D}_i + K_i$, where

$$\hat{D}_i = \hat{G}_i + \int_0^A \hat{D}^*_{i-1}(x)\hat{p}(x)dx \qquad (6.21)$$

and

$$K_i = zC_i + K^*_{i-1}\,. \qquad (6.22)$$

We minimize the value of S_i to get the optimal available asset value A^*_i for this stage.[32] From equation 4.31, we get the optimal OC:

$$C^*_i = A^*_i - L - K^*_{i-1}\,. \qquad (6.23)$$

The optimal total capital is $T^*_i = C^*_i + K^*_{i-1}$. We also have optimal values of the components $\hat{D}_i$,

[31] For example, suppose a three-period model has a standard deviation (SD) of 50 for the first period loss increment, 60 for the second period and 80 for the third period. This process will provide optimal results for the three-period case and will also give the optimal results for a one-period model with an 80 SD, and a two-period model with a 60 SD for the first period and 80 for the second period.

[32] Appendix C discusses the optimization technique, which uses two asset values whose difference is small.

and K_i, which we label $\hat{D}_i^*$ and K_i^*. So now we have the three inputs needed to determine the optimal capital for the stage $i + 1$, and successive stages, until the optimal initial capital for the Nth period is found.

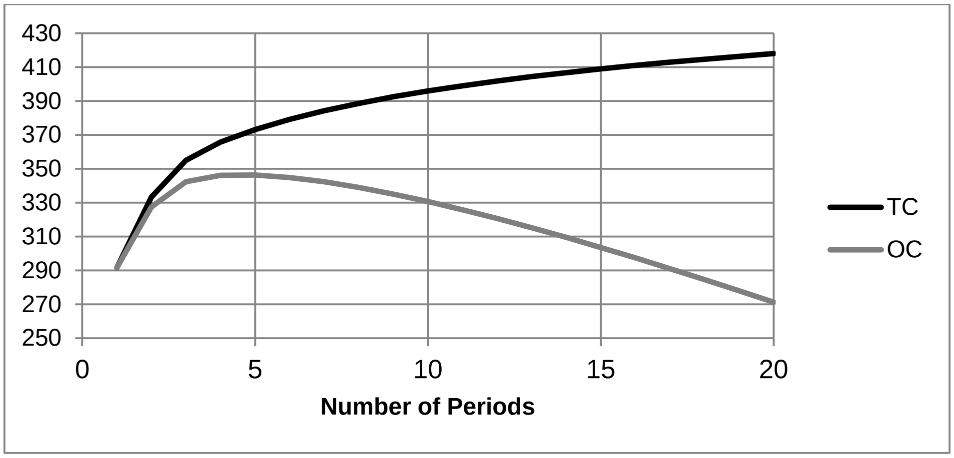
To illustrate this process, consider the basic section 5 example from Appendix B. We have $\hat{D}_1^*$ = 0.7852 (from Appendix B.1) and K_1^* = 5.8325 (from Appendix B.3). Applying equations 6.21 and 6.22, and iterating gives the following optimal values for a time horizon ranging from 1 through 4 periods.

Table 6.21
Optimal Values by Number of Periods
Normal-Exponential Example; AC Strategy

No. of Periods	Initial Total Capital	CED	Capital Cost	Ownership Capital
1	291.62	0.7852	5.8325	291.62
2	333.34	1.5309	12.3827	327.51
3	354.95	2.2367	19.2317	342.45
4	365.70	2.9212	26.1537	346.10

Extending the example to 20 periods, Figure 6.21 compares optimal initial total capital (TC) and ownership capital (OC) amounts by period length.

Figure 6.21
Optimal Initial Total and Ownership Capital Amount by Number of Periods
Normal-Exponential Example; AC Strategy



Notice that the optimal initial total capital increases steadily, but at a declining rate, as the number of periods increases. Therefore, as the ultimate loss variance increases, the optimal initial total capital also increases.

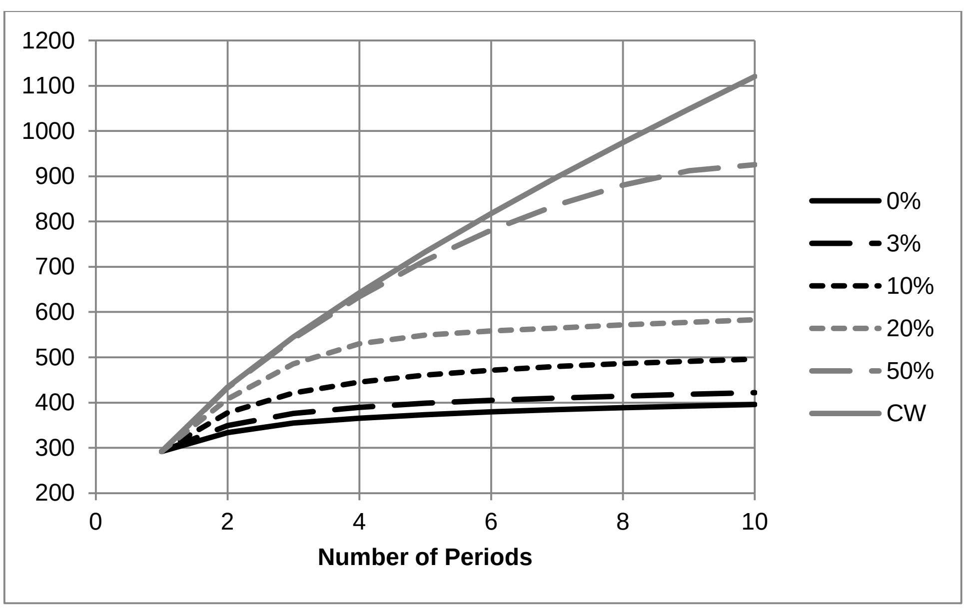
However, the pattern *for ownership capital* is different and rather interesting: for a small number of periods (5 in this example) the optimal initial OC increases, and then decreases with a longer horizon. Eventually, with a long enough horizon (17 periods here) the optimal initial OC is *less* than that for a single period. The reason for the declining amount of OC is that the premium component of the expected future capital costs provides additional assets in excess of the owner-supplied capital; the policyholder-supplied capital increases faster with horizon length than the amount of total capital needed to offset default risk.

6.3 Multi-Period Capital with ACR Strategy

As shown in section 5.42, the two-period optimal ACR capital calculation requires *two* optimal capital amounts at each stage: one based on the cost of holding capital z and a smaller amount based on z plus w, the cost of raising capital. Appendix D develops the recursive relationships needed for the optimal ACR initial capital for *N* periods. Here we need *six* optimal quantities at each stage: three similar to those in section 6.2 (based on no CRC) and three more based on the higher capital costs under the CRC.

Also, since incorporating the CRC creates loss region 2b (where capital remains the same for the next period), an additional calculation is required: at each stage, the expected CED and capital cost for this region must be found by numerical integration. Figure 6.31 extends the section 5.42 example to 10 periods and shows the optimal initial capital for *w* ranging from 0% to 50%. It also shows the optimal initial total capital for the CW strategy, which effectively has an infinite cost of raising capital.

Figure 6.31
Optimal Initial Total Capital Amount by Number of Periods
and Cost of Raising Capital
ACR Strategy; Normal-Exponential Example



7. CAPITALIZATION INTERVAL

The preceding analysis has used an arbitrary period length, with capital flows occurring at the beginning of each period. Since the period length governs the duration between capital flows, and to distinguish it from other insurance periods such as policy term, I specifically refer to the period length as the *capitalization interval* (abbreviated as CI).

The actual length of the CI will affect the optimal capital, since, for a given loss duration, a shorter capitalization interval will allow more opportunities to add or withdraw capital as the loss amount evolves. To analyze this effect, recall that the policy term is defined to be equal to the period length. Thus, the losses occur at the beginning of the policy term,[33] and capital flows also occur at the beginning and end of the policy term. Section 7.3 discusses the case where the period length is shorter than the policy term.

The frequency of potential capital additions and withdrawals will have a significant impact on the optimal capital and solvency cost, regardless of the capital strategy used.

7.1 Capitalization Interval with the FR Strategy

To illustrate the effect of the CI, again assume the basic one-period normal example from section 5 with a standard deviation of 100. The optimal total capital is 291.62 with a solvency cost of 6.62. The period length and loss duration are both *one year*: thus, capital is supplied at the beginning of the year and the amount of loss is known at the end of the year. Also assume that the stochastic process is continuous over time: for every smaller period the loss variance is proportional to the period length. Now suppose that we subdivide the one-year period into half-year periods, with capital flows allowed at the beginning of each. Each smaller period will now have a loss standard deviation of

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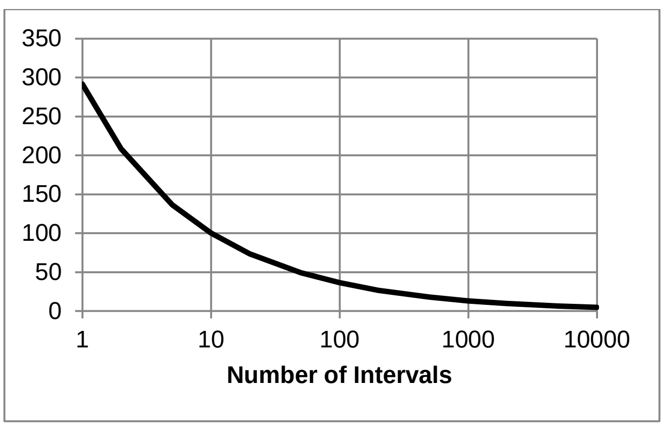
[33] A more realistic assumption is that the loss may occur randomly throughout the policy term, with the average loss happening at the middle of the term. Here, I am merely attempting to show the effect of changing the capitalization interval length. A practical application would use the actual expected timing of the incurred losses.

$70.71 = 100 / \sqrt{2}$ and the capital cost rate is 0.01 = 0.02/2. Under the full recapitalization (FR) strategy, the optimal beginning total capital for each half-year period is now 208.56 with a corresponding 2.34 solvency cost. The solvency cost for the entire year is twice this amount, or 4.68. Thus, by allowing more frequent capital movement, the consumer value has improved and less capital is required.

Figure 7.11 shows the effect of further subdividing the one-year period into more capitalization intervals:

Figure 7.11
Optimal FR Total Capital by Number of Intervals
One-Year Loss Duration
Normal-Exponential Example

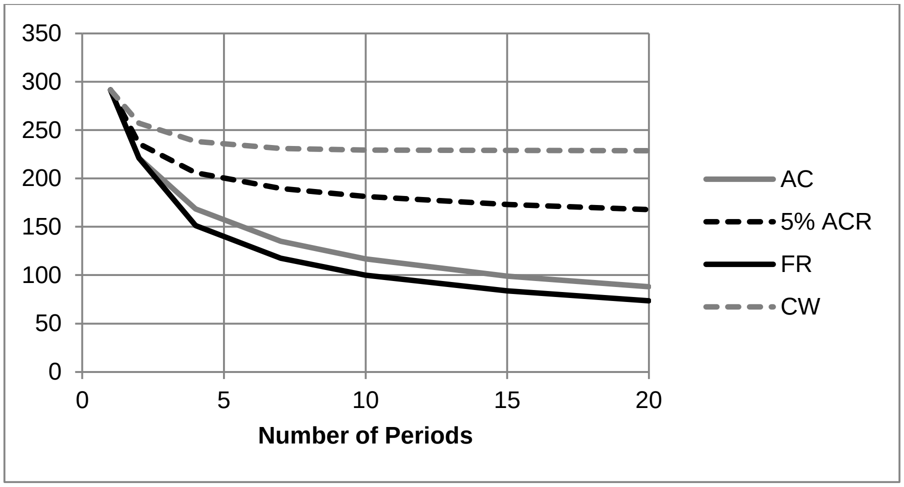


As the number of intervals becomes large, the optimal capital amount approaches zero! Although not shown in this graph, the annual solvency cost associated with the optimal capital also approaches zero (it is only 0.096 for 10,000 intervals). Since capital is added in response to infinitesimal changes in loss evaluation, there is only a tiny chance at any time that a default will occur, and if it does, the default amount will be infinitesimally small. Notice that this result depends on the assumption of a *continuous* distribution for the loss increment: if the loss valuation can change in somewhat large increments, then the capital additions cannot "catch up." Consequently, in a theoretical world with a continuous stochastic loss process and the ability to add capital with no cost, there is no need for an insurer to carry capital. However, the loss process might not be continuous, and, as discussed next in section 7.2, important real-world imperfections, frictions and costs do not permit an infinitesimally small CI, so capital is indeed required.

7.2 Capital Strategies and Time Intervals

For other capital strategies, the optimal capital also declines as the CI becomes smaller. Figure 7.21 compares results of the FR, AC, CW and ACR (with 5% cost of raising capital) strategies, according to interval length. Here, I show the *average* amount of total capital over the year. Note that the initial capital for the first period will also decline with the number of intervals for the FR, AC and ACR with low capital-raising cost strategies. However, for the CW and high capital-raising cost ACR strategies, the first-period capital amount *increases* with the number of intervals. Nevertheless, since these strategies stockpile capital in the early periods and tend to withdraw more of it later than for the other strategies, the average amount of capital declines with the number of intervals.

Figure 7.21
Optimal Average Total Capital For One Year
by Number of Intervals and Capital Strategy
Normal-Exponential Example



Here, all strategies provide smaller optimal capital amounts and solvency costs as the CI length decreases. The AC strategy follows the FR strategy in that the optimal capital approaches zero as the interval approaches zero. However, the optimal capital for the CW and ACR strategies declines much more slowly[34] because either capital cannot be added, or its addition is costly.

Although the optimal capital and solvency cost decline with shorter period length, there will be a practical limit to this effect. Even for a pure continuous stochastic loss process, the minimum interval length is governed by real-world considerations. The minimum length depends on a sequence of events, each of which requires some time. Among other factors, the loss reserve must be evaluated (for most insurers this occurs monthly or quarterly) and then management must decide to raise capital and then contact an investment bank. The bank then performs due diligence and offers the public an opportunity to supply capital. Even if the insurer has a prior commitment from

[34] It appears that the average capital may reach a fixed limit, but I have not proved this.

an investment bank, this process may take several months.

Nevertheless, it is clear that policies with short capitalization intervals will require less capital than longer ones, and will be more efficient (with lower solvency costs) as well.[35] Because some insurers may be better-equipped to generate capital flows quickly, the minimum interval length will vary by insurer. Consequently, this factor should be considered in assessing specific insurer capital levels.

7.3 Effect of Policy Term and Capitalization Interval

The preceding analysis has assumed that the capitalization interval equals the policy term. Generally, the policy term for property-casualty insurance is one year,[36] but the capitalization interval will most likely be shorter than one year. Assume that the premium is paid at the beginning of the period. When the capitalization interval is shorter than the policy term, insolvency may occur early in the policy term. This event will effectively terminate coverage for losses that may occur in the remainder of the policy term, and will produce an additional solvency cost, since the full premium is paid up front.

Here, more capital is needed if the policy term exceeds the CI, regardless of the number of intervals. Thus, besides the CI, which greatly affects the optimal capital, the *length of the policy term* is another variable that will influence the capital amount. This effect will be present with both short-duration and long-duration[37] losses, since the cost of foregone coverage must be considered.

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[35] Because the ability to quickly raise capital decreases solvency costs, the capitalization interval length will also affect short-duration losses in a manner similar to that of long-duration losses: optimal capital is less if the CI length is shortened.

[36] Some automobile policies have a six-month term and, less commonly, some commercial risks have multi-year coverage.

[37] Modeling this effect is more complicated than for property, since one must assume a relationship between the losses of each interval. For liability coverage, these will be correlated. A convenient approach is to assume that all losses move together.

8. MULTI-PERIOD MODEL EXTENSIONS

This section extends the basic loss model to incorporate features that may be necessary for a practical application. Also, I briefly discuss how the results might apply to life insurance.

8.1 Stochastic Time Horizon

In a more realistic model of the development process for long-duration losses, the ultimate duration of losses is not known. Here I relax the basic model assumption that the loss develops randomly for N periods and is settled at the end of the Nth period. Instead, assume that, although the value evolves according to the section 4.2 liability stochastic process, the process may terminate randomly at the end of each period, at which point the loss is settled. In this model, there are N possible periods, extending to the longest possible claim duration. Call this model the *stochastic-horizon* (abbreviated as SH) loss model.

Let q_i represent the probability of settlement at the end of period *i*. Then $q_1 + q_2 + \cdots + q_N = 1$. From section 4.21, with a constant per-period loss volatility the variance of the ultimate loss will be $\sigma^2[1 \cdot q_1 + 2 \cdot q_1 + \cdots + N \cdot q_N]$, since the variance of each period's loss increment is independent of the prior value. This is a simple weighted average of the loss variances of the component N possible models.

Meanwhile, assume that the certainty-equivalent expected value of the SH loss is proportional to the variance (as discussed in section 4.42). Consequently, the CE expected value of the SH loss must equal the weighted average of the CE expected loss values of its N component loss models, where the weights are the termination probabilities q_i.

Under the SH model, the optimal capital will be a weighted average of the optimal capital values for the component fixed-horizon models. Given the above analysis, to approximate the optimal SH capital, it is reasonable to use the *exact termination probabilities* (rather than a CE adjusted set of probabilities) to weight the optimal capital amounts.

To illustrate the SH model, we extend the basic normal-exponential example. Assume three periods with termination probabilities q_1 = 0.5, q_2 = 0.3 and q_3 = 0.2. The average loss duration is 1.7 periods and the respective expected amounts of loss paid at the end of each period are 500, 300 and 200. From section 6.2, the optimal initial total capital amounts for the three component horizons are T_1^* = 291.62, T_2^* = 333.34 and T_3^* = 354.95. Thus, the optimal initial capital for the basic SH model is 316.80 = 0.5(291.62) + 0.3(333.34) + 0.2(354.95).

Notice that, if the expected loss is independent of the loss duration (as in the basic model) the set of termination probabilities will represent the expected loss payment pattern.

8.2 Interest Rates and Present Values

Because multi-period losses, especially for liability insurance, can be paid several years from when the loss occurs, it is necessary to use the *present value* of the solvency cost components in determining optimal capital. Since the present value of a certainty-equivalent amount must also be a CE value, the present value is found using a *risk-free* interest rate, denoted by a rate *r* per period. A similar logic applies to the capital cost component. Note that this assumption of a single rate implies a flat yield curve; a practical application might require a separate riskless rate for each component duration.

For a one-period model, the present value of the solvency cost is

$$S_1 = (zC + \hat{D}) / (1 + r) . \tag{8.21}$$

The initial assets equal the capital plus the premium. The premium is $\pi = (L + zC) / (1 + r)$, which equals the present value of the expected loss and capital cost components. At the end of the period, before the loss and capital costs are paid, the initial assets grow to $C(1+r)+L+zC$. The capital cost zC is expended prior to the loss payment, so the assets available to pay the loss are $A=C(1+r)+L$. Thus $\partial C / \partial A = 1/(1+r)$. Also, from section 2.2 we have $\partial \hat{D} / \partial A = -\hat{Q}(A)$. The optimal solvency cost is found from $\partial S_1 / \partial A = 0$, giving

$$\hat{Q}(A) = z/(1+r). \tag{8.22}$$

The amount $z/(1+r)$ is the *calibration level* of the risk measure $\hat{Q}(A)$. With $r = 0$, we have $\hat{Q}(A) = z$, the result used in the earlier sections of this paper. With zero interest, we find the optimal assets satisfying the calibration level and subtract the expected loss amount to get the optimal capital. With a positive r, however, equation 8.22 gives the *ending* available assets; subtracting the expected loss gives the ending optimal capital, which must be reduced by a factor of $1 + r$ to produce the optimal *initial* capital. Consequently, the optimal initial capital is the *present value* of the amount of capital required with a zero interest rate, where the calibration level equals the present value of the capital cost rate.

Besides affecting the present value of the optimal capital, the interest rate level will affect the frictional cost of capital (through double-taxation), as discussed in EBRM. I assume that the expected default has a negligible impact on the premium. If the insurer's income tax rate is t, the frictional cost of capital component due to double taxation for one period equals $rt/(1-t)$ times the capital amount. It is useful to separate the frictional cost of capital into two components: the double-taxation cost, which depends on r and the other costs z_0 (such as financial distress and regulatory restriction costs) that do not depend on *r*. We then have

$$z = rt/(1-t) + z_0 . \tag{8.23}$$

If $r = 0$, then $z = z_0$ and the calibration level is z_0. If $r > 0$, the calibration level is greater than z_0 for $z_0 < t/(1-t)$. This inequality will hold for plausible values of t and z_0.[38] Therefore, the optimal capital amount from equation 8.22 will be *less than* the present value of the zero-interest optimal capital, since the calibration level is lower (making capital more costly).

To illustrate the effect of the interest rate on optimal capital, I modify the basic section 5.3 example for one period. Assume that z_0 = 0.5% and t = 30%. With $r = 0$, we have z = 0.5% from equation 8.23, giving a 0.5% calibration level and optimal capital of 347.59. Increasing r to 5% boosts the calibration level to 2.52% and the optimal capital drops to 267.69. This amount is significantly less than the present value of zero-interest optimal capital: 331.08 = 347.59/1.05.

The above analysis shows that the interest rate level reduces optimal capital (from that with zero interest) in two ways. First, there is a present value effect: since the initial capital grows at the rate *r*, less capital is needed to offset a potential default occurring in the future. Secondly, the frictional cost of capital is greater due to double-taxation on the increased investment income from capital: since the cost of capital is greater, insurers will use less of it.

For a multi-period model, interest rates can readily be incorporated by modifying the section 6 backward induction method. To illustrate, I use the basic AC strategy with no cost of raising capital.

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[38] The value of z_0 is likely to be on the order of magnitude of 1%. Even if the insurer's effective income tax rate is as low as 10% (giving $t/(1-t)$ =11.1%), the relationship holds.

Let $S_i(r)$ denote the present value of the solvency cost for *i* periods with interest rate *r*.

Using the section 6 backward induction indexing, define the present value of the expected capital cost as $K_i(r)$, which is the analog of K_i in equation 6.22:

$$K_i(r) = [zC_i + K_{i-1}^*(r)] / (1+r) . \qquad (8.24)$$

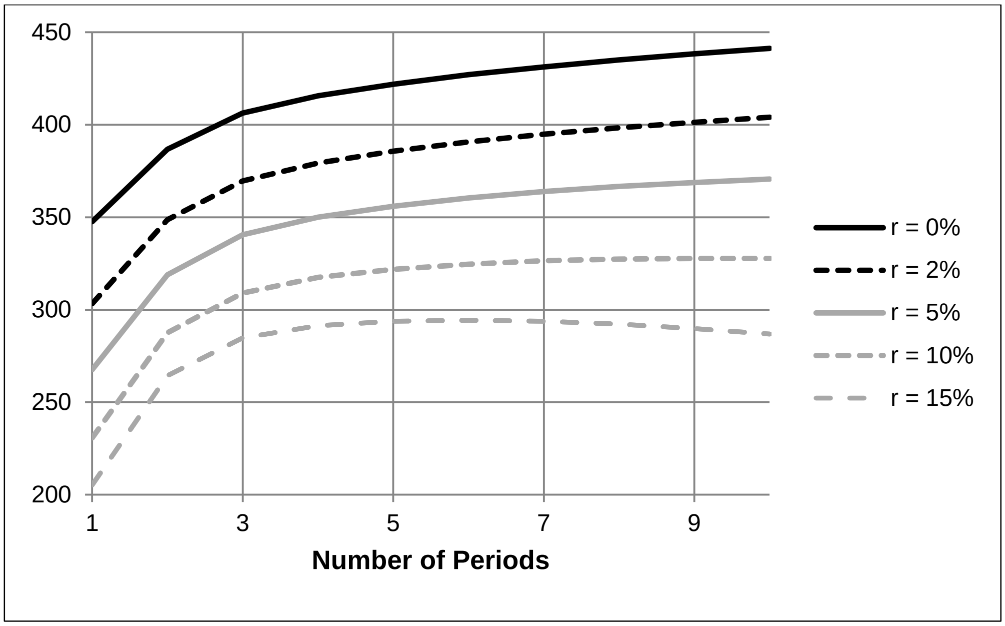
The solvency cost for stage *i* is therefore

$$S_i(r) = \hat{D}_i(1+r)^{-i} + K_i(r) . \qquad (8.25)$$

The default is realized at the end of *i* future periods, so its CE expected value $\hat{D}_i$ is discounted for *i* periods. However, the capital costs are discounted for a shorter time span on average, since they occur over the entire horizon length.

Starting from the one-period optimal CED present value $\hat{D}_1^* / (1+r)$ and optimal capital cost present value $K_1^*(r)$, we use equation 8.25 recursively to generate the successive optimal capital amounts. To demonstrate this calculation, I use the basic AC example from section 6. Figure 8.21 compares the optimal total capital for *r* ranging from = 0%, to 15%, for horizons of one to ten periods.

Figure 8.21
Optimal Initial Total Capital by
Riskless Interest Rate
Normal-Exponential Example with AC Strategy



Notice that, as for the single-period case, optimal capital is *less* for a given time horizon if the interest rate increases. Indeed, since $z_0 < t/(1-t)$, the optimal capital is also less than the present value of the zero-interest optimal capital for each time horizon.

Also, for large interest rates, the optimal initial total capital *decreases* with the horizon length beyond a certain point. The transition occurs at 10 periods with r = 10% and 7 periods with r = 15% in the above example. This happens because the average duration of the capital costs (roughly i/2) is less than that for the default duration i.

8.3 Risk Margins

The preceding analysis assumed that the loss component of the premium included only the unadjusted expected value of the loss, i.e., the premium did not reflect a positive market price for bearing the risk. Here I assume that the market value of the insured loss, denoted by $\overline{L}$, is greater than the expected loss: i.e., it contains a *risk margin*, whose value is denoted by *M*. The expected market-value loss, including the risk margin, can be determined from a *third* stochastic process with an adjusted probability distribution. For a one-period model, we have

$$\overline{L} = \int_0^\infty \overline{p}(x) x dx = L + M , \tag{8.31}$$

where $\overline{p}(x)$ denotes the adjusted probability underlying the risk margin. Here the relevant risk is systematic: it cannot be reduced through pooling, and therefore commands a price in financial markets. The value to the policyholder of the underlying risk, before it is reduced through insurance pooling, will be larger per unit of expected loss than that of the insurer's risk (which is larger than the expected loss). So we have $\hat{L} \geq \overline{L} \geq L$.

For a multi-period stochastic process with equal variance of loss increments for each period, I assume that the risk margin increases uniformly with the number of periods. In that case, if the stochastic process is additive, then the risk margin will also be additive. Let *m* represent the risk margin as a ratio to the expected loss *L* for one period. So, for an *N*-period loss, the risk margin will equal *mLN,* and the market value of the expected loss will be *L*(1 + *Nm*).

For a multiplicative stochastic process, the market value of *L* is $L(1+m)^N$. Observe that the present value of the market-value loss is $L(1+m)^N(1+r)^{-N}$, where *r* is the risk-free interest rate. Therefore, the market value of the expected loss can be expressed as the expected value *L*, discounted at a risk-adjusted interest rate $r_a = (r-m) / (1+m)$,[39] where $r > r_a$.

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[39] Butsic [1988] develops the risk-adjusted interest rate for insurance reserving and pricing applications.

The fair premium is $\pi = L + M + K$. The risk margin then provides additional total capital, beyond the amount from the expected capital cost *K* (see section 6.2). The amount *M* can also be considered as policyholder-supplied capital; to give the same insolvency protection, the insurer will need less ownership capital than without the risk margin. As discussed in section 4.4, the risk margin is equivalent to ownership capital in terms of solvency protection.[40] For multiple periods, the relationship holds as well, since the CED depends on the asset amount and not the accounting measure of the loss. To illustrate this effect, assume the basic AC normal model with a 0% interest rate, and let m = 2%.[41] From section 5.31, for a one-period model without the risk margin, the optimal total capital is 291.62 and optimal available assets are 1291.62. The expected default depends on the asset level, not the capital amount as defined by the accounting method. With the risk margin, the same assets are also optimal: the CED is the same, and changing the asset amount through the initial owner-supplied capital will reduce the consumer value.

Thus, the premium and initial assets will be larger by 20 = 0.02(1000). Optimal one-period capital is now reduced by 20 to 271.62 to give the same CE default probability (equal to the capital cost rate). Since the risk margin in this example is proportional to the number of periods, the optimal initial capital is reduced by 20 units times the number of periods in the time horizon.

Figure 8.31 compares the optimal initial ownership capital for the AC strategy by time horizon

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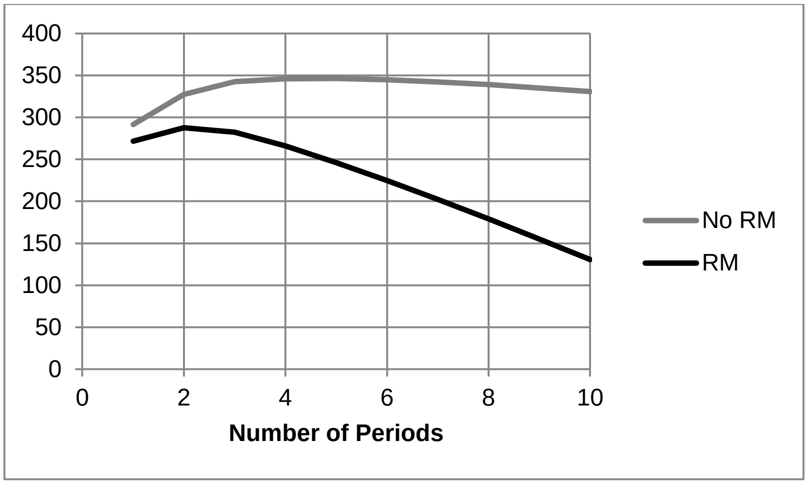
[40] In another sense, the risk margin may be considered as *ownership* capital in that it is not a third-party obligation: it "belongs" to the owners of the insurance firm and will be returned to the owners if the insurance proves to be profitable. However, depending on the accounting method used for income taxation, the risk margin may not generate a frictional capital cost (it currently does not in the U.S.).

[41] In practice, the amount of risk margin is a small fraction of premium. It is straightforward to show that the risk margin is $m = [(R - r) / (1 - t)] / [C / L]$, where *R* is the expected return on the capital *C*, *r* is the risk-free return and *t* is the income tax rate. For example, assume an insurer's current expected (after-tax) return on equity is about 4% above the risk-free investment return, the effective tax rate is 30% and the leverage ratio C/L is 40%. A risk margin equal to 2.3% of expected loss will provide the required return on equity. Note also that the risk margin cannot exceed the risk value: the difference between the CE value of the loss and its expected value; otherwise the policyholder is better off without insurance.

for no risk margin and for a 2% risk margin.

Figure 8.31
Optimal Initial Ownership Capital by Time Horizon
2% Risk Margin (RM) vs. No Risk Margin (No RM)
Normal-Exponential Example with AC Strategy



The optimal ownership capital without a risk margin here is the same as in figure 6.21. Note that the *total* capital (also shown in Figure 6.21) continues to increase with the time horizon.

In sections 5 through 7, I have assumed that the expected capital cost should be determined from an unadjusted loss distribution. However, with a risk margin, the expected capital cost for future periods should be calculated using *the market value* loss distribution $\overline{p}(x)$ since the future capital amount depends on the future random loss value. The expected capital cost is larger, compared to the no-risk margin case, and the optimal fair-value capital will be less. Nonetheless, for simplicity, I have ignored the effect of the market value loss distribution on capital costs for this section.

8.4 Life Insurance

This paper has focused on property-casualty insurance. As such, the scope of the study precludes a thorough development of optimal capital for life and health insurance. However, below I briefly discuss some implications of the findings in this paper to life insurance products (note that health insurance is similar to property-casualty insurance in that policy terms are short and there are few embedded options).

Life Insurance Liability Risk

Generally, for life insurance the risk of losses being higher than expected is low due to the lack of correlation between claims from separate policies. There is some chance of default from losses occurring earlier than expected (e.g., whole life insurance) or later than expected (e.g., annuities). The risk of default for the amount of claims and their timing can be addressed by the techniques presented in the earlier sections. Life claims risk has a different stochastic process than long-duration losses, since the periodic indemnity amounts are fixed but the horizon is stochastic. The process is not Markovian, since if more/fewer insureds die, then the probability of future deaths changes for the insured population.

Embedded Policyholder Options

A major source of risk for life insurers is the nature of the embedded options in policy contracts. These are not usually present for property-casualty insurance. For example, life policyholders may stop paying premiums or they may add coverage after the policy has been in force; policyholders may be able to make loans at favorable terms; the policy may have other investment guarantees. The effect of any of these depends on policyholder behavior. Note that some policy features may not remain after the insurer becomes insolvent and is under conservatorship. Moreover, the policy features that create default risk have value to the policyholder, which should be incorporated into the consumer value in the optimal capital calculation.

Capital Funding Strategies

Notwithstanding the above differences between life and property-casualty insurance, the capital funding strategies available to life insurers are the same as for property-casualty insurers. The availability and cost of external capital will also be similar. These factors will have parallel impacts on the amount of capital needed for life insurers. Also, modeling the asset risk will be similar, since both types of insurers have the same categories of investments in their portfolios.

9. MULTI-PERIOD ASSET RISK

The preceding results for risky losses can be extended to the case where *assets* are risky. Here I develop a method for integrating asset risk into the model and show some basic results for the AC strategy.

I start with a treatment of asset risk in a one-period model that differs from the method in EBRM. In that analysis, I assumed the certainty-equivalent ending value of risky assets equaled the terminal value of the assets as if they were invested in risk-free securities. A better assumption is that the CE value of the risky assets equals their unadjusted expected ending value minus a quantity (called the *risk premium* in financial economics) that mirrors the *risk value* (as defined in section 2.23) for losses. The results from this analysis are consistent with standard finance techniques for optimizing an individual's investment portfolio.[42]

9.1 One-Period Joint Loss and Asset Model

For one period, where both the loss and the ending asset amount are random, the CE expected default value is

$$\hat{D} = \int_0^\infty v\hat{p}(v)\,dv\,. \tag{9.11}$$

Here v represents the difference between the loss and available asset values for all combinations

[42] See Bodie, Kane and Marcus [2014], chapter 6.

of loss and asset values that produce *v*, with $\hat{p}(v)$ being the CE probability of *v* occurring. The expression for the unadjusted expected default D is similar to that equation 9.11, with the unadjusted probability *p*(*v*) replacing $\hat{p}(v)$.

Assume that the insurer's assets consist of riskless securities having a zero return, as specified in section 4.1, as well as an amount of risky assets *AR* with an expected (market) rate of return r_M per period. The risky assets are diversified and have the same standard deviation (SD) of return σ_M as the market rate of return. Thus, if the insurer's ending asset value has a SD of $\sigma_A = AR\sigma_M$, then its expected return amount is $ER = \sigma_A(r_M / \sigma_M)$, which equals the asset risk SD times the Sharpe ratio.[43] Notice that, although the Sharpe ratio is commonly applied to stock market risk, it can also characterize bond market risk: the expected return on a long-term bond will normally exceed that of a short-term Treasury note; meanwhile the bond value has a positive volatility due to potential interest rate fluctuations.

The variance of *v*, or the total variance, is $\sigma_T^2 = \sigma_L^2 + 2\rho\sigma_L\sigma_A + \sigma_A^2$, where σ_L^2 is the loss variance and ρ represents the correlation between loss and asset values.[44] Also assume that the insurer maintains a *constant asset risk* as it changes its capital amount, so that additions and withdrawals are in riskless assets.

In equation 9.11 (either the CE or unadjusted version), the expected default amount can be

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[43] The ratio of the expected excess return (over the risk-free rate) on a security to the standard deviation of the return is known as the Sharpe ratio. Here the risk-free rate is assumed to be zero, so the Sharpe ratio is simply equal to r_M / σ_M.

[44] For some insurance products that depend on investment performance (such as embedded options in life insurance and property-casualty products where the market value of losses depend on interest rates), the co-variation with losses may be more complex than we can represent with a simple correlation coefficient. Thus, more extensive modeling may be required for a practical application.

obtained by assuming that the asset risk is *zero* and the original loss distribution is replaced by an alternative loss distribution that produces the same default amounts with the same probabilities as the joint asset and loss distribution. Call the alternative loss variable the *augmented* loss. Since the expected ending asset amount (prior to paying the loss) is greater than the initial value by the expected return *ER*, the expected augmented loss, denoted by L_A, is reduced by this amount; thus $L_A = L - ER$. Accordingly, the augmented loss has mean L_A and standard deviation σ_T.

Since the sum of two jointly distributed normal random variables is normal, if both the loss and ending asset amounts are normally distributed, then the augmented loss variable is also normal. If the asset and loss variables are not normal, then the augmented loss technique will produce approximate results. The subsequent analysis in this section assumes joint normality for the two variables.

The expected certainty-equivalent default calculation is the same as that of a risky loss with mean L_A and variance σ_T^2, together with riskless assets. To illustrate, assume that policyholder risk aversion is based on exponential utility. We have $\hat{L} = L + a\sigma_L^2 / 2$ (from section 4.22), and $\sigma_T^2 = \sigma_L^2 + 2\rho\sigma_L\sigma_A + \sigma_A^2$. Thus the certainty-equivalent expected augmented loss is

$$\hat{L}_A = L_A + a\sigma_T^2 / 2 = \hat{L} + RP - ER. \tag{9.12}$$

The quantity $RP = a\rho\sigma_L\sigma_A + a\sigma_A^2 / 2$ denotes the asset *risk premium*. In finance this represents the amount by which the expected return is reduced to produce the CE ending value of the assets. If the ending asset values and losses are statistically independent, then $\rho = 0$, giving $\sigma_T^2 = \sigma_L^2 + \sigma_A^2$ and $RP = a\sigma_A^2 / 2$.

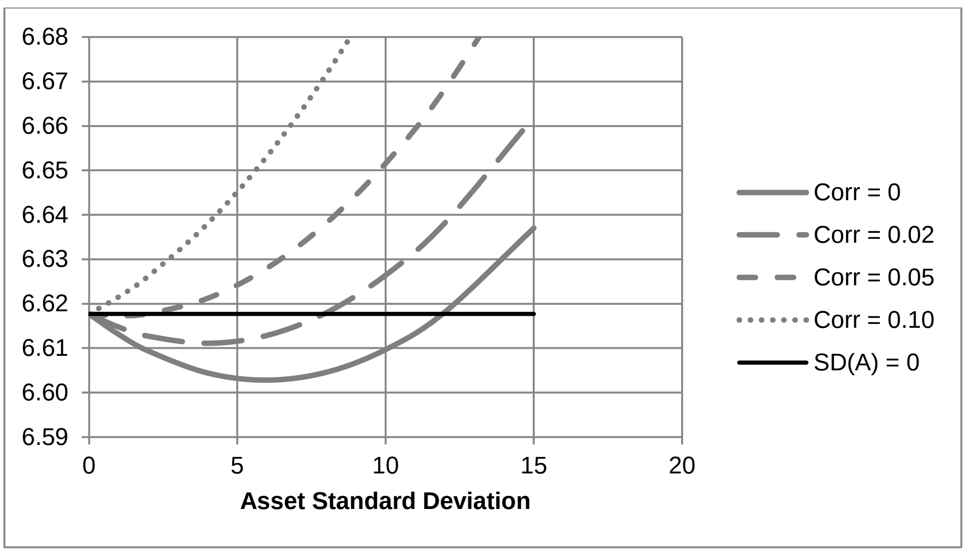
Using the augmented loss and total variance, the optimal capital for an insurer with both asset and loss risk can be determined in the same way that we calculate the optimal capital for an insurer with riskless assets in a one-period model. By subtracting the optimal joint capital from the optimal capital for riskless assets, we get the implied optimal amount of capital for the risky assets.

Because the expected return on risky assets is positive, including some risky assets in the insurer's investment portfolio can *reduce* the solvency cost compared to that from a riskless portfolio. To illustrate this, we use the basic example from section 5.3. Suppose that, instead of riskless assets, the insurer now has AR = 50 units of risky assets with the remainder in risk-free securities. The market expected return is r_M = 5% with a volatility of σ_M = 20% and the assets are uncorrelated with losses. Thus, the insurer's asset risk is $\sigma_A = 10$ = 0.2(50), the expected return amount is ER = 2.5 = 50(0.05), the risk premium is $RP = 0.50 = 0.01(10)^2/2$ and the total risk is σ_T = 100.50. Equation 9.12 gives $\hat{L}_A$ = 1048 = 1050 – 2.5 + 0.5. The optimal capital is then 291.02, which is *less* than the 291.62 with no asset risk. The solvency cost with the risky assets is 6.6097, which is also less the 6.6177 for riskless assets. In this case, the optimal capital for asset risk is *negative*: –0.60 = 291.02 – 291.62. This example shows that a moderate amount of asset risk can actually improve policyholder welfare: a situation akin to an individual benefitting from having an investment portfolio containing some risky securities.

As the amount of risky assets increases, the expected return increases linearly with σ_A, but the risk premium increases with its square. Consequently, the beneficial effect of the expected return will vanish if the asset risk is too high. Also, the asset risk is mitigated by its combination with the loss risk if the two are independent: if there is adverse correlation[45] (where high/low loss values tend to correspond with low/high asset values), then the benefit of the expected return is also reduced.

Figure 9.11 shows these two effects. Here I show the solvency cost for the above example by asset risk amount with correlation values of 0, 0.02, 0.05 and 0.10. The horizontal line indicated by SD(A) = 0 is the optimal solvency cost without asset risk.

Figure 9.11
Optimal Solvency Cost by Asset Standard Deviation
And Asset/Loss Correlation
One-Period Normal-Exponential Example



All points below the zero asset-risk line represent situations where risky assets will improve solvency cost and all points above the line indicate a portfolio that worsens the solvency cost. Notice that in this example, if $\rho = 0$, then any amount of asset risk less than a standard deviation of

[45] A mechanism for this effect is the Fisher hypothesis, where inflation and interest rates tend to move in tandem. This co-movement can increase unpaid loss values while depressing bond and stock values. Notice that the correlation coefficient is positive in this case, since the joint variance is greater under adverse correlation than for independence.

about 12 (i.e., risky assets are less than about 5% of the 1291 in total assets) will improve the solvency cost. If the asset SD is 5.95, the optimal solvency cost is attained. Also notice that if the asset/loss correlation is above approximately 0.06, then no amount of risky assets will improve the solvency cost.

Although a small amount of asset risk is optimal in this example, the solvency cost is not far from optimal if asset risk is moderately higher. For example, if the insurer has risky assets with a 40 SD, the optimal solvency cost is 7.08, which is 0.46 greater than the zero asset-risk optimum of 6.62. However, the difference represents only about 0.05% of the expected loss, so this reduction of value to the policyholder may not be material in a practical setting. The optimal capital for this case is 311.34, which is greater than the 291.62 with no asset risk; the 19.71 difference represents the amount of capital needed for asset risk.

For a one-period insurer model, if the assets are bonds whose market values have a low correlation with the insurance losses, the above analysis shows that under a normal (positive-sloping) yield curve, the optimal portfolio will have a duration to maturity that exceeds a single period. Thus, in assessing capital adequacy, the standard actuarial technique of matching asset and liability durations may not produce optimal results.

9.2 Multi-Period AC Joint Loss-Asset Model

Extending the one-period joint loss-asset model to two or more periods is relatively straightforward as long as the asset risk can be incorporated into the loss as in equation 9.12. If not, a more complex numerical method or simulation may be necessary.

For a single-period model, the riskless interest rate (which I have assumed to be zero) is known at the beginning of the period. However, for more than one period, the future interest rate will vary randomly, with a mean of zero. I assume that the distribution of asset returns exceeding the risk-free rate is independent of the level of the rate. Thus, the certainty-equivalent expected default values in equation 9.12 and their counterparts for multiple periods will be the same as if the future riskless rates were fixed at the initial one-period value. The present value of the expected CE default and the expected capital costs (see section 8.2) can be determined by using the expected risk-free rate,[46] which is zero for the simplified model in this section.

For longer horizons, joint loss-asset risk is not quite parallel to the case of multi-period risky losses. The loss value will continue to evolve if technical solvency occurs at the end of the first period, as in the loss-only model. However, with risky assets, if the insurer becomes technically insolvent after the first period, the asset risk will drop to virtually zero since the insurer will enter conservatorship shortly after becoming technically insolvent. As discussed in section 3.2, the asset portfolio will be converted to an essentially riskless one by the conservator. For simplicity, assume that the investment portfolio is immediately converted to riskless assets upon technical insolvency. We also maintain the constant asset-risk assumption from the one-period model: if the insurer remains solvent, the asset portfolio retains the same risk as the size of the portfolio changes.

Therefore, the optimal capital calculation under the joint loss-asset model is the same as with the loss-only model having the total risk σ_T, except that (1) the available assets are greater by the amount of the expected return on assets ER and (2) the CE technical default amount $\hat{G}$ is based on only the loss risk σ_L.

To illustrate long-horizon asset-risk capital, we can extend the one-period example from section 9.1 to the range of one to ten periods, using the AC capital strategy. Here the asset risk is 10, 20 or

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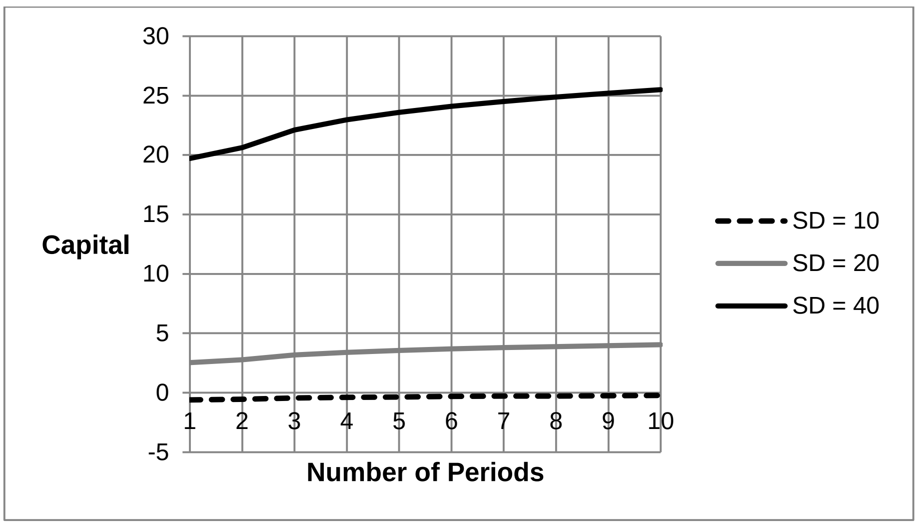
[46] In theory, the best rate for discounting these expected cash flows is the risk-free spot yield matching the length of the cash flow. With a normal yield curve, the spot yield for several periods will be greater than that of a single period. Although I have ignored this feature in developing the basic multi-period model, it can easily be incorporated into a practical application.

40 per period with a zero asset/loss correlation. We calculate the optimal joint total capital for each horizon length and for each asset risk amount. The optimal total asset-risk capital is the difference between the optimal joint total capital and the optimal total capital without risky assets. Figure 9.21 displays these results.

Figure 9.21
Optimal Initial Total Asset-Risk Capital by Time Horizon
And by Asset Standard Deviation
Normal-Exponential Example with AC Strategy

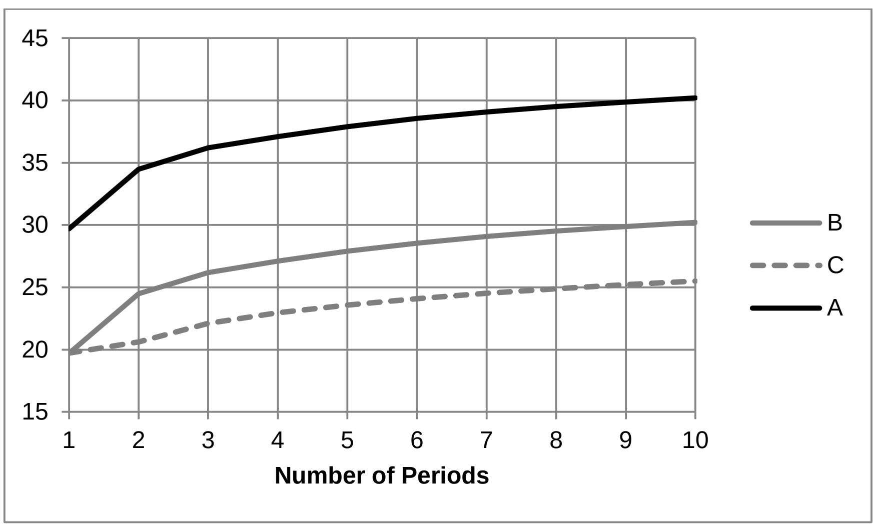


Notice that the optimal asset-risk capital for an asset risk SD of 10 is negative for each horizon length, just as it is for a single period. Also, the optimal amount of asset-risk capital increases slightly with the horizon length for each asset risk SD.

It is interesting to show the effect of the two major elements of asset risk that differentiate asset risk capital from loss risk capital: the expected return from risky assets and the elimination of risky assets under technical insolvency. We start by assuming that there is no expected excess (of the risk-free) return for risky assets. In this instance, labeled Case A, the optimal capital is the same as that from loss-only risk having the same SD as the total joint loss and asset risk, or σ_T . We next assume in Case B, that there is a positive excess return for risky assets. In Case C (which represent the model in figure 9.21), we assume both a positive excess return and that assets are converted to riskless securities if a technical insolvency occurs.

Assume that the asset risk SD is 40 (with an expected return of 10). Figure 9.22 shows the optimal first-period asset-risk total capital for horizons of one to ten periods, for each of the above three cases.

Figure 9.22
Optimal AC Initial Total Asset-Risk Capital Comparison
by Time Horizon and by Asset Assumptions (A, B and C)
Section 9.2 Example; Asset SD of 40



The presence of the expected return (Case B) reduces the optimal asset-risk capital from that of Case A by the amount of expected return for a single period, or 10 in this example. The amount of expected return acts as another source of default-reducing assets, such as the policyholder-supplied capital (built into the premium) for risk margins and frictional capital costs. Notice that the policyholder-supplied capital is provided only once, at the time the premium is written, covering all subsequent periods. In contrast, the expected return effectively provides added capital for each period on an ongoing basis as long as the insurer is solvent and maintains the asset portfolio.

Eliminating asset risk when insolvent (Case C) reduces the asset-risk capital further for multi-period horizons, as long as the amount of asset risk is greater than the optimal amount (an SD of 5.95 per period). If the asset risk is *lower* than the optimal amount, then the elimination of asset risk when insolvent will slightly increase the optimal asset-risk capital for each period.

10. CONCLUSION

The purpose of this study is to determine, in principle, the risk-based capital for multi-period insurance losses and assets. Using basic economic concepts central to insurance, I have shown how to find the optimal multi-period capital amount without arbitrarily choosing which risk measure (e.g., VaR, TVaR and others) and time horizon model (one-year vs. runoff) should be used. The analysis gives proper weight to volatility in each period and incorporates important constraints, such as conservatorship under technical insolvency and the ability to raise capital externally. Much of this undertaking is new territory. In particular, the notions of policyholder risk preferences and dynamic capital strategies may be unfamiliar to an actuarial audience. While falling short of a full practical application, I have provided numerical examples to illustrate how the concepts might be applied if the underlying parameters are known.

The major qualitative results of this paper are summarized in section 1.1. Perhaps the chief among them are: (1) the optimal capital for long-horizon losses depends on *both* the annual loss volatility and the ultimate loss volatility, and will be greater than optimal capital based on the annual volatility, and (2) optimal capital for any horizon depends on the insurer's ability to raise capital, and its cost of raising capital. Analyzing the first relationship is largely a technical actuarial exercise, while analyzing the second involves understanding an insurer's connections to capital markets, ownership structure and internal information processes.

Knowing the optimal capital provides the basis for applications in product pricing, corporate governance and regulation. Due to the many variables involved, optimizing capital for multi-period insurance can be rather complicated and perhaps daunting. However, as shown here, starting from a basic one-period model, the requisite multi-period model can be assembled step-by-step to produce useful results. More extensive modeling with additional elements can be accomplished using simulation techniques.

The analysis in this paper has identified some important factors relevant to multi-period risk that are not commonly considered in setting capital standards for insurance: capital funding strategies, the cost of raising external capital, the capitalization interval, policy term, ownership structure and the effect of conservatorship. These topics provide a fertile source for future research.

APPENDIX A: EXAMPLES FOR DISCRETE STOCHASTIC PROCESS

Section 4.2 Example

The loss stochastic process can be illustrated with a simple two-period binary example. The initial expected loss is 1000 and the reserve increments X_1 and X_2 each can be either 200, or –200 with probability 0.5, giving a per-period variance of $(200)^2$. Let the risk value per period be 100. Then we have $\hat{L}_1 = 1100$ and $\hat{L}_2 = 1200$. The first period CE expected loss of 1100 is obtained by assigning a CE probability of 0.75 to the +200 reserve increment and 0.25 to the –200 increment.

The evolution of the ultimate loss and its certainty-equivalent counterpart is shown in Figure 4.221 below. The first-period reserve increment probabilities and CE probabilities are denoted by p_1 and $\hat{p}_1$, with p_2 and $\hat{p}_2$ representing the second-period values.

Figure 4.221
Loss Reserve Evolution, Binary Numerical Example

L	p_1	$\hat{p}_1$	L_1	p_2	$\hat{p}_2$	L_2	$p_1 \times p_2$	$\hat{p}_1 \times \hat{p}_2$
				0.5	0.75	1400	0.25	0.5625
	0.5	0.75	1200					
				0.5	0.25	1000	0.25	0.1875
1000								
				0.5	0.75	1000	0.25	0.1875
	0.5	0.25	800					
				0.5	0.25	600	0.25	0.0625
Expected Value			**1000**			**1000**		
CE Expected Value			**1100**			**1200**		

Notice that for each period the variance of the loss increment is the same and that the variance of the evolved loss increases over time. Meanwhile, the mean for each subsequent period equals the value of the loss from the prior period: for instance, if L_1 becomes 1200 at the end of period 1, then 1200 is the mean for period 2. The CE expected value of the second-period loss conditional on the emerged 1200 amount is 1200 plus the 100 risk value for the second period, or 1300.

Section 5.1 Example

Assume that the expected loss is 1000 and increments for each period range from –400 to 400 in steps of 50; the corresponding loss probabilities are generated by a binomial distribution having a base probability 0.5 with 16 trials. Thus the probability of a 400 increment is $(0.5)^{16}$, the probability of a 350 increment is $16(0.5)^{16}$, and so forth. The expected value of the increments is zero and the variance is $(100)^2$. For the parallel CE stochastic loss process, assume that the base probability is 0.625, giving a higher subjective likelihood of larger increments: the probability of a 400 increment is $(0.625)^{16} = 0.00054$ and the probability of a 350 increment is $16(0.625)^{15}(0.375) = 0.00520$. The CE expected value of the increment is 100, so the CE expected loss increases by a risk value of 100 each period.

Now suppose that initial assets are 1300, so a technical insolvency occurs if the first-period loss is either 1350 or 1400 (the maximum possible loss). When the technical insolvency occurs, the assets remain fixed at 1300, but the loss can still develop for one more period. Consequently, if the first-period loss is 1350, its value at the end of the second period is one of $\{1350 - 400, 1350 - 350, \cdots, 1350 + 400\}$, or $\{950, 1000, \cdots, 1750\}$. However, only the amounts $\{1350, 1400, \cdots, 1750\}$ will produce a default when the loss is settled at the end of the second period. The respective CE probabilities for these amounts are $\{0.11718, 0.17361, \cdots, 0.00054\}$. Weighting the possible default amounts by their occurrence probabilities gives 152.59, the conditional CED given that the 1350 loss amount occurs.

For the 1400 first-period loss, the range of its possible second-period values that produce an ultimate default is larger: from 1350 to 1800. Thus, its conditional CED is larger, at 200.72, than that for the 1350 loss amount. Table 5.11 outlines these calculations.

Table 5.11
Conditional Certainty-Equivalent Expected Default
Two-Period Numerical Example
Binomial Stochastic Process; Assets = 1300

One-period Loss	CE Probability (a)	0.00054	0.00520	. . .	0.06250	0.02625	Total
1400	2P Loss (b)	1800	1750	. . .	1400	1350	
	Default (c): [(b) – 1300]	500	450	. . .	100	50	
	CE Expected Default: [(a) x (c)]	0.27	2.34		11.72	3.12	**200.72**
1350	2P Loss	1750	1700	. . .	1350		
	Default	450	400	. . .	50	0	
	CE Expected Default	0.24	2.08	. . .	5.86	0	**152.59**

The unconditional CED is determined by weighting the above conditional amounts by the CE probabilities of the 1350 and 1400 loss values occurring. We get

$\hat{G}$ = **0.9029** = 0.00054(200.72) + 0.00520(152.59). Notice that under the FR strategy, with the same 1300 in initial assets, the technical insolvency at the end of the first period is converted to a hard insolvency. So the CED equals the possible default amounts (50 = 1350 – 1300 and 100 = 1400 – 1300) multiplied by the respective CE probabilities: $\hat{D}$ = **0.3144** = 0.00520(50) + 0.00054(100). For comparison with the FR strategy, notice that for each loss value producing a default (e.g., 1350) the default amount (50 here) is fixed under FR, but will further develop under AC (the CE expected value is 152.59).

APPENDIX B: NORMAL-EXPONENTIAL MODEL

B.1 Optimal One-Period Results

From EBRM (Appendix A4), if risk aversion is based on exponential utility with risk-aversion parameter *a*, and the loss distribution is normal with mean L and standard deviation σ, then we have

$$\hat{Q}(A) = \frac{Q(A)}{Q(A) + P_s(A)/Y} \tag{B.11}$$

and

$$\hat{D} = -\ln[P_s(A) + YQ(A)]/a. \tag{B.12}$$

Here $P_s(\,)$ represents the cumulative normal probability with the shifted mean $L_s = L + a\sigma$ and standard deviation σ. Also, $Y = e^{a(A-\hat{L})}$.

Equation B.11 is used to determine optimal capital for one period. To illustrate the optimal capital calculation, let L = 1000, σ = 100, a = 0.01 and z = 0.02. For one period, we have $\hat{L} = L + a\sigma^2/2$ = 1050, so Y = 11.203. Since $\hat{Q}(A) = z$, equation B.11 gives optimal assets A = 1291.62, thus optimal capital is 291.62. From equation B.12, the optimal CED is $\hat{D}^*$ = 0.7852.

B.2 CE Value of Technical Default for Multiple Periods

Equation B.12 is needed to determine the value of $\hat{G}$, the CED under technical insolvency for two or more periods. If the time horizon is N periods, and the insurer becomes technically insolvent at the end of the first period, then $N-1$ periods remain. For each loss outcome L_1, the CE expected ultimate loss is $\hat{L}_1 = L_1 + (N-1)a\sigma^2/2$. The conditional CED value is readily found from equation B.12, and the unconditional value of $\hat{G}$ equals the sum of the conditional CED amounts, weighted by their CE probabilities of occurrence.

For example, if we have a three-period model with assets A = 1400 and the first-period loss value is L_1 = 1500, the insurer is technically insolvent. Two periods remain; the loss now has a mean value of 1500 and will develop to its ultimate amount over the two periods. We have $\hat{L}_1 = L_1 + (N-1)a\sigma^2 / 2$ = 1600 and a normal standard deviation of 141.42 = $100\sqrt{2}$. Accordingly, equation B.12 gives the CE default $\hat{D}$ = 216.10 for this particular loss outcome. Using numerical integration,[47] we weight this value and the other CED amounts for losses exceeding 1400, by the corresponding CE probabilities of the losses, to get $\hat{G}$ = 0.1809.

B.3 Optimal Two-Period AC Capital Example

To illustrate the optimal two-period capital calculation, we extend the above example to two periods with σ = 100 for each period. Using equation 5.214, we adjust the available asset level A, until the minimum solvency cost is attained. This occurs when $\hat{G} =$ 0.7514, $\hat{H} =$ 0.7794 and K = 12.3827. Thus, the optimal solvency cost is

S = 13.9136 and the optimal initial total capital is 333.34. This amount is greater than the 291.62 needed for a one-period model with the same first-period variance.

Notice that if we start with the optimal capital amount for one period (developed above), we have A = 1291.62, giving $\hat{G} =$ 2.1325, $\hat{H} =$ 0.7695, K = 11.5483 and

S = 14.4503. This result is sub-optimal, so more capital is needed.

Under the FR strategy for two periods, the optimal total capital for the first period is also 291.62. However, the ownership capital is less than 291.62 by the amount of the expected second-period

[47] For these calculations, I used 1,000 discrete ending first-period loss values to approximate the result.

capital cost (which is policyholder-supplied capital contained in the premium) of K_1^* = 5.8325, so the first-period OC equals 285.79. The optimal solvency cost is 13.1187 = 2(0.7852) + 5.8325+ 0.02(285.79).

APPENDIX C: SECTION 5.42 EXAMPLE

Suppose that the cost of raising capital is w = 3% and initial assets are 1400. Thus we get C_R^* = 246.50 (from equation 5.411) and C^* = 291.62. We need to determine the expected cost of the capital and the CE value of the expected default. Assume that the initial first-period total capital is 400. Table 5.422 shows these solvency costs by region.

Table 5.422
Expected CE Default and Second-Period Expected Capital Cost by Region
Two-Period AC Strategy with 3% Cost of Raising Capital
Normal-Exponential Example
Initial Total Capital of 400

Region	Exp. CE Default	Exp. Capital Cost
1	0.0130	0.0002
2a	0.4538	0.3884
2b	0.1661	0.4169
3	0.5349	5.0204
Total	1.2578	5.8258

The expected CED for region 1 is the technical default amount; for region 2a it is the expected CED corresponding to C_R^*, times the CE probability that the loss is in the region; for region 2b it equals the sum of all second-period CED values weighted by the CE loss probabilities (using numerical integration). The region 3 expected CED equals the expected CED corresponding to C^*, times the CE probability that the loss is in the region. The expected capital costs are determined in a parallel fashion. However, for region 2a, the expected amount of capital raised is 2.6964, so the 0.3884 amount includes the 3% cost of raising capital, or 0.0809. Also, since I have assumed for simplicity that capital for region 1 is still required after technical insolvency, the expected capital cost is very small, at 0.0002.

The first-period capital cost is 8.0000 = 0.02(400), and so the total solvency cost for both periods is 15.0836 = 8.000 + 5.8258 + 1.2578. To obtain the optimal value of the solvency cost, I perform a parallel calculation with a small increment (0.01) to the initial capital. Using a value of 400.01, the solvency cost differs by 0.000124. The capital is optimal when the difference is zero, so by iterating with different initial capital amounts, the optimal value is 349.04. At that point the solvency cost is 14.7102.

APPENDIX D: BACKWARD INDUCTION WITH ACR STRATEGY

Under the ACR strategy, there are two optimal capital amounts to consider at each stage *i* of the iteration. The first is the optimal OC given the current loss value is *small enough* to withdraw capital. This is the amount $C_{i-1}^{*}(x)$ defined under the AC strategy. The second is the optimal capital $CR_{i-1}^{*}(x)$ given the current loss value is *large enough* to add capital (by raising it externally).

At each stage *i*, there are now six optimal quantities that we need to calculate: the three from the AC strategy (capital, CED and capital cost), and their counterparts given that capital is raised: the optimal capital is defined above, the optimal CED of $\hat{D}R_{i}^{*}(x)$ and the optimal capital cost $KR_{i}^{*}(x)$. Since the *i*th period capital cost is not included, the corresponding optimal total capital is $TR_{i}^{*}(x)=CR_{i}^{*}(x)+KR_{i-1}^{*}(x)$.

At each stage, these three capital-raising components are found by using a capital cost for the current period of *z* + *w* instead of only *z*. We then have a parallel calculation of the solvency cost $SR_{i}=\hat{D}R_{i}+KR_{i}$, which is minimized by changing the asset amount.

Also, at each stage it is necessary to calculate the CED and capital cost components for region 2a (where capital is neither raised nor withdrawn) by numerical integration: we vary the capital amount in this region and weight the results by the corresponding loss probabilities.

To illustrate this process, I use the normal-exponential example with a 3% CRC. For one period we have the key variables C_1^* = 291.62, $\hat{D}_1^*$ = 0.7852, K_1^* = 5.8325,

CR_1^* = 246.50, $\hat{D}R_1^*$ = 2.2839 and KR_1^* = 4.301. To obtain the optimal two-period value C_2^*, we start with an arbitrary initial capital amount (the optimal one-period capital of 291.62 is a good start) and calculate the solvency cost as in Appendix C. This is done by adding the CED and capital cost components for the four regions of first-period loss outcomes (see section 5.42). This calculation uses the above six key variables. We perform a parallel calculation with the capital increased by a small amount (say, 0.001). We adjust the capital amount (and its incremental counterpart) until the difference between the incremental and the original solvency costs is zero. This occurs when A = 1349.04 and S_2 = 14.7102, giving T_2^* = 349.04, C_2^* = 343.16,

$\hat{D}_2^*$ = 1.8536 and K_2^* = 12.8566.

We next do a second calculation where the first period capital cost is $z + w = 0.05$. This provides the optimal values of the key variables for the case where capital is raised after the first period of a three-period horizon (we are preparing for the next stage of the induction procedure). Here we get

TR_2^* = 301.00, $\hat{D}R_2^*$ = 3.2764 and

KR_2^* = 12.0567.

We continue the induction process to get the optimal key variables for longer horizons.

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Society (CAS) in preparing risk-based capital proposals for the National Association of Insurance Commissioners. I joined the CAS RBC Dependency and Correlation Working Party, led by Allan Kaufman. As my contribution to this effort, I began a project to determine the best solvency risk measure for property-casualty insurers. The paper *An Economic Basis for Property-Casualty Insurance Risk-Based Capital Measurement* was the product of that effort. I also undertook a second project, to determine risk-based capital for multi-period insurance. This assignment greatly expanded my earlier work, and this paper is the result. I am deeply thankful for Allan's patient stewardship in guiding me along this lengthy learning process. His innumerable astute comments, sharp critique and editorial suggestions were invaluable; they forced me to explain results more clearly — the paper is much better for his involvement.

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GLOSSARY OF ABBREVIATIONS AND NOTATION

Abbreviation	Meaning	Section Defined
AC	Add capital (strategy)	4.5
ACR	Add capital (strategy) with cost of raising capital	5.4
CE	Certainty-equivalent	2.1
CED	Certainty-equivalent expected default	2.1
CI	Capitalization interval	7.3
CW	Capital withdrawal (strategy)	4.5
EBRM	Economic Basis … Risk Based Capital Measurement	1
FA	Fixed assets (strategy)	4.5
FCC	Frictional capital cost	2.2
FR	Full recapitalization (strategy)	4.5
IFRS	International Financial Reporting Standards	1
OC	Ownership capital	4.4
SD	Standard deviation	9.1
SH	Stochastic horizon	8.1
VaR	Value-at-risk	1
TVaR	Tail value-at-risk	1

Variable	Meaning	Section Defined
a	Risk aversion parameter	4.2
A	Assets	2.1
AR	Risky asset amount	9.1
C	Capital (ownership)	4.1
CF	Capital flow	4.5
CR	Capital raised externally	App. D
D	Expected default	2.1
DR	Expected default if capital is raised	App. D
$E(\,)$	Expectation operator	4.3
EF	Expected capital carried forward	5.4
ER	Expected return	9.1
G	Expected default under technical insolvency	5.1
H	Expected default for remaining periods	5.2
i	Period index	4.3
K	Expected capital cost	4.3
KR	Expected capital cost if capital is raised	App. B
L	Expected loss	2.1
M	Risk margin value	8.3
N	Number of periods	4.1
$p(\,)$	Probability density	2.1
$P(\,)$	Cumulative probability	6.2
q	Probability of period length	8.1
Q	Default probability	2.2
r	Risk-free interest rate	8.2

r_M	Market rate of return	9.1
R	Expected return on capital	8.3
RP	Risk premium	9.1
S	Solvency cost	5.2
SR	Solvency cost with capital raised externally	App. D
t	Income tax rate	8.3
T	Total capital	4.4
TR	Total capital when raised externally	App. D
v	Loss minus asset value	9.1
V	Consumer value	2.2
w	Cost of raising capital	5.4
x	Loss or asset size	2.1
X	Reserve increment	4.2
Y	Ratio of successive reserve amounts	4.2
z	Frictional cost of capital	2.2
∂	Partial derivative operator	5.2
Δ	Capital increment	App. B
π	Premium	2.2
ρ	Asset/loss correlation	9.1
σ	Standard deviation	4.2
Subscript		
a	Region 2a	5.4
A	Assets	9.1
b	Region 2b	5.4
E	Ending capital	5.4
L	Losses	9.1
M	Market	
R	Raising capital	5.4
t	Elapsed time	4.2
T	Total assets and losses	9.1

BIOGRAPHY OF THE AUTHOR

Robert P. Butsic is a retired actuary currently residing in San Francisco. He served as a member of the American Academy of Actuaries Property-Casualty Risk Based Capital Committee and is a member of the Casualty Actuarial Society's Risk-Based Capital Dependency and Calibration Working Group. He previously worked for Fireman's Fund Insurance and CNA Insurance. He is an Associate in the Society of Actuaries, has a B.A. in mathematics and an MBA in finance, both from the University of Chicago. He has won the Casualty Actuarial Society's Michelbacher Award (for best Discussion Paper) five times. Since the 2008 financial crisis he has enjoyed reading economics blogs, which have contributed to the development of this paper.